Multigrid Methods for the Computation of Singular Solutions and Stress Intensity Factors: Corner Singularities

Seminar "Elliptic Problems on Non-Smooth Domains"
JKU Linz WS 2013/14

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References

S. Brenner. Multigrid Methods for the Computation of Singular Solutions and Stress Intensity Factors I: Corner Singularities. Math. of Comp., 68:226 p559-583, 1999

Outline

Outline

1. Introduction
2. Formula for Stress Intensity Factors κ
3. Two Multigrid Methods

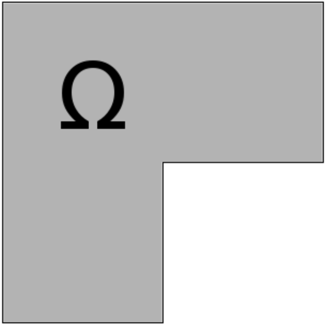
Introduction

Model Problem

Let Ω be bounded polygonal domain and $f \in L^2(\Omega)$. Consider

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

- Here: Ω Γ-shaped (for simplicity)
- Note: All results generalize to polygonal domains with more than one re-entrant corners



Recap (Talk 26/11/2013 by P. Gangl)

$$\Delta : V^2 + X \to L^2(\Omega) = \mathcal{R}(\Delta; V^2) + \mathcal{R}(\Delta; V^2)^\perp \text{ surjective}$$

- $V^2 = H^2(\Omega) \cap H_0^1(\Omega)$ (Dirichlet case)
- $X = \underset{0<\lambda_{j,m}<1}{\text{span}} \{S_{j,m}\}$
- $S_{j,m}(r_j, \theta_j) = \eta_j(r_j)\, r_j^{\lambda_{j,m}}\, \varphi_{j,m}(\theta_j)$
  - $\eta_j(r_j)$ ... cut-off function
  - $\lambda_{j,m}$, $\varphi_{j,m}$ such that

$$-\varphi''_{j,m}(\theta_j) = \lambda_{j,m}\varphi_{j,m}(\theta_j)$$

Dirichlet case: $\lambda_{j,m} = \frac{m\pi}{\omega_j}$ $\qquad \varphi_{j,m}(\theta) = \sqrt{\frac{2}{\omega_j}}\sin(\theta\lambda_{j,m})$

  - $S_{j,m} \in H^1(\Omega)$ for $\lambda_{j,m} \geq 0$
  - $S_{j,m} \notin H^2(\Omega)$ for $0 < \lambda_{j,m} < 1$
  - $S_{j,m} \in H^r(\Omega)$ for $r < 1 + \underset{j,m}{inf}\{\lambda_{j,m} : 0 < \lambda_{j,m} < 1\}$

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$$\Delta : V^2 + X \to L^2(\Omega) = \mathcal{R}(\Delta; V^2) + \mathcal{R}(\Delta; V^2)^\perp \text{ surjective}$$

Image Space:

- $L^2(\Omega) = \mathcal{R}(\Delta; V^2) + \mathcal{R}(\Delta; V^2)^\perp$ because $\mathcal{R}(\Delta; V^2)$ closed
- $N := \mathcal{R}(\Delta; V^2)^\perp = \underset{0<\lambda_{j,m}<1}{\text{span}} \{\sigma_{j,m}\}$
- $u_{j,m}(r_j, \theta_j) = \eta_j(r_j)\, r_j^{-\lambda_{j,m}}\, \varphi_{j,m}(\theta_j)$

Dirichlet problem on Γ-shaped domain:

- Dirichlet Problem: $0 < \lambda_{j,m} = \frac{m\pi}{\omega_j} < 1 \iff m = 1 \wedge \omega_j > \pi$
- Γ-shaped domain: only 1 non-convex corner with angle $\omega (= \frac{3\pi}{2})$
- $s_+(r, \theta) :=$ "$S_{j,m}(r_j, \theta_j)$" $= \eta(r)\, r^{\pi/\omega} \sin(\frac{\pi}{\omega}\theta)$
- $s_-(r, \theta) :=$ "$u_{j,m}(r_j, \theta_j)$" $= \eta(r)\, r^{-\pi/\omega} \sin(\frac{\pi}{\omega}\theta)$

Recap (Talk 26/11/2013 by P. Gangl)

$$\Delta : V^2 + X \to L^2(\Omega) = \mathcal{R}(\Delta; V^2) + \mathcal{R}(\Delta; V^2)^\perp \text{ surjective}$$

Theorem

Let Ω be a bounded, polygonal, open subset of $\mathbb{R}^2$. For each $f \in L^2(\Omega)$ there exists a unique $u \in V := H_0^1(\Omega)$ (variational) solution of

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{dx} = -\int_\Omega f \, v \, \mathrm{dx}$$

for every $v \in V$ and there exist unique numbers $c_{j,m}$ such that

$$u - \sum_j \left(\sum_{0<\lambda_{j,m}<1} c_{j,m} S_{j,m} \right) =: w \in H^2(\Omega).$$

On Γ-shaped domain: $u = w + u_S = w + \kappa \, s_+$

κ ... stress intensity factors

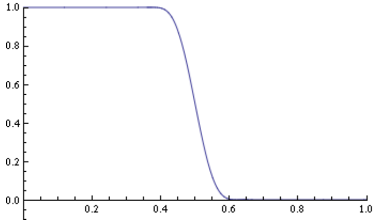
Recap (Talk 26/11/2013 by P. Gangl)

Some functions:

- $\eta(r) = \begin{cases} 1 & r \leq r_0 \\ (1 + e^{\psi(r)})^{-1} & r_0 < r < r_1 \\ 0 & r_1 \leq r \end{cases}$

  $\psi(r) = \frac{1}{2}\left(\frac{1}{r_1 - r} - \frac{1}{r - r_0}\right)$

  Note: $\eta \in C^\infty(\mathbb{R})$



$\eta(r)$

Relations:

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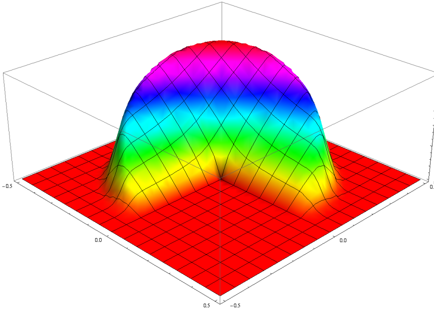
- $s_+(r,\theta) = \eta(r) r^{\pi/\omega} \sin(\pi/\omega\theta)$



Relations:

$s_+(r,\theta)$

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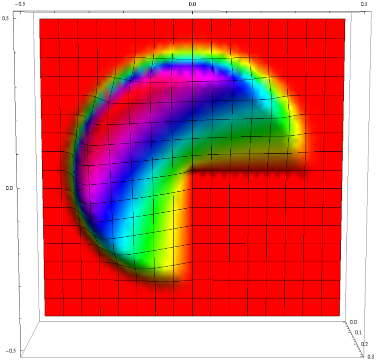

$s_+(r,\theta)$

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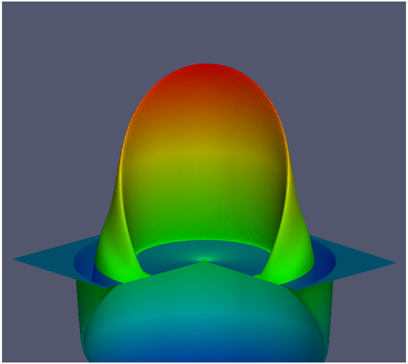

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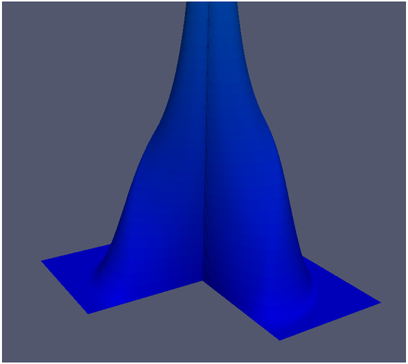

$s_-(r,\theta)$

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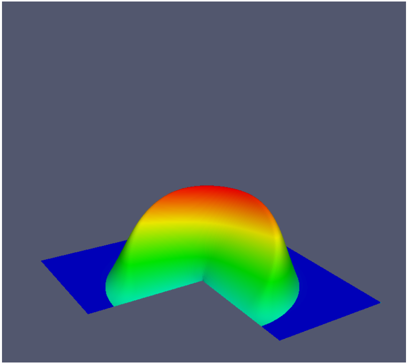
- $s_-(r, \theta) = \eta(r) r^{-\pi/\omega} \sin(\pi/\omega\theta)$
- $v_{j,m}$ $(\in H^1(\Omega))$



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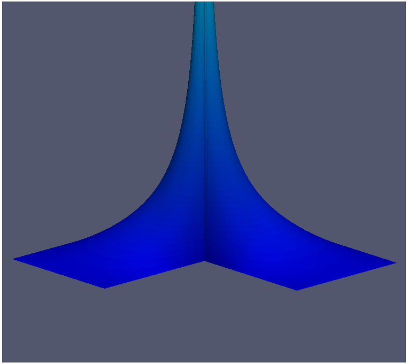
- $v_{j,m}$ $(\in H^1(\Omega))$
- $\sigma_{j,m}$ $(\in N)$

Relations:



$\sigma_{j,m}$

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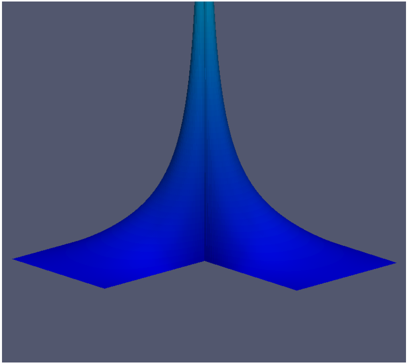
- $v_{j,m}$ $(\in H^1(\Omega))$
- $\sigma_{j,m}$ $(\in N)$

Relations:

- Δs_+ not orthogonal to $N = \text{span}\{\sigma_{j,m}\}$



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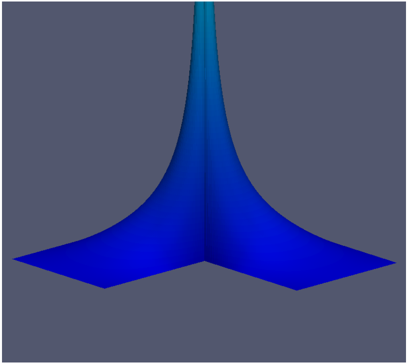

$\sigma_{j,m}$

Relations:

- Δs_+ not orthogonal to $N = \text{span}\{\sigma_{j,m}\}$
- $\sigma_{j,m} = s_- - v_{j,m}$

Outline

Calculation of Coefficient κ

For a Γ-shaped domain we have

$$\begin{aligned} u &= w + u_S \\ &= w + \sum_{0<\lambda_{j,m}<1} \kappa_{j,m} S_{j,m} \\ &= w + \kappa\, s_+ \end{aligned}$$

with $w \in H^2(\Omega) \cap H^1_0(\Omega)$ and $s_+(r,\theta) = \eta(r)\, r^{\pi/\omega} \sin(\frac{\pi}{\omega}\theta)$.

Question: $\kappa = ?$

Calculation of Coefficient κ

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Question: $\kappa = ?$

If κ was known: Consider

$$\begin{aligned} -\Delta w &= f + \kappa \Delta s_+ && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega \end{aligned}$$

$\Longrightarrow$ Good convergence rate (since $w \in H^2(\Omega)$)

Calculation of Coefficient κ

Consider

$$\int_{\Omega} (u\Delta s_- - s_-\Delta u)\,\mathrm{d}x$$

Calculation of Coefficient κ

Consider

$$\int_\Omega (u\Delta s_- - s_-\Delta u)\,\mathrm{d}x = \underbrace{\int_\Omega (w\Delta s_- - s_-\Delta w)\,\mathrm{d}x}_{(*)} + \underbrace{\int_\Omega (u_S\Delta s_- - s_-\Delta u_S)\,\mathrm{d}x}_{(**)}$$

Calculation of Coefficient κ

Regular part (∗):

$$\int_\Omega (w \Delta s_- - s_- \Delta w)\, \mathrm{d}x$$

Calculation of Coefficient κ

Regular part $(*)$:

$$\int_\Omega (w\Delta s_- - s_- \Delta w)\,\mathrm{d}x$$

Recall: Lemma

There exists $\sigma \in N$ such that

$$\sigma - \underbrace{\eta(r)\, r^{-\pi/\omega} \sin((\pi/\omega)\theta)}_{=:s_-} =: -v \in H^1(\Omega).$$

Recall (proof): $v = R\,[(-\Delta)s_-]$ with $R : H^{-1}(\Omega) \to H_0^1(\Omega)$ Riesz isomorphism

Calculation of Coefficient κ

Regular part $(*)$:

$$\int_\Omega (w\Delta s_- - s_-\Delta w)\,\mathrm{d}x$$
$$= \int_\Omega (w\Delta s_- - R[(-\Delta)s_-]\Delta w)\,\mathrm{d}x - \underbrace{\int_\Omega (s_- - R[(-\Delta)s_-])\Delta w\,\mathrm{d}x}$$

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$$= \int_\Omega (w\Delta s_- + \nabla R[(-\Delta)s_-]\cdot\nabla w)\,\mathrm{d}x$$

Recall: Lemma

There exists $\sigma \in N$ such that

$$\sigma - \underbrace{\eta(r)\, r^{-\pi/\omega}\sin((\pi/\omega)\theta)}_{=:s_-} =: -v \in H^1(\Omega).$$

Recall (proof): $v = R\,[(-\Delta)s_-]$ with $R : H^{-1}(\Omega) \to H^1_0(\Omega)$ Riesz isomorphism

Calculation of Coefficient κ

Regular part $(*)$:

$$\begin{aligned}
&\int_\Omega (w\Delta s_- - s_- \Delta w)\,\mathrm{d}x \\
&= \int_\Omega (w\Delta s_- - R[(-\Delta)s_-]\Delta w)\,\mathrm{d}x - \underbrace{\int_\Omega (s_- - R[(-\Delta)s_-])\Delta w\,\mathrm{d}x}_{=0} \\
&= \int_\Omega (w\Delta s_- + \nabla R[(-\Delta)s_-]\cdot\nabla w)\,\mathrm{d}x = 0.
\end{aligned}$$

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There exists $\sigma \in N$ such that

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Recall (proof): $v = R\,[(-\Delta)s_-]$ with $R : H^{-1}(\Omega) \to H_0^1(\Omega)$ Riesz isomorphism

Calculation of Coefficient κ

Singular part $(**)$:

$$\int_\Omega (u_S \Delta s_- - s_- \Delta u_S)\,\mathrm{d}x = \kappa \int_\Omega (s_+ \Delta s_- - s_- \Delta s_+)\,\mathrm{d}x$$

Calculation of Coefficient κ

Singular part (∗∗):

$$\int_\Omega (u_S \Delta s_- - s_- \Delta u_S)\,\mathrm{d}x = \kappa \int_\Omega (s_+ \Delta s_- - s_- \Delta s_+)\,\mathrm{d}x$$

Since

$$\Delta s_\pm = \left[\eta''(r) + \left(1 \pm \frac{2\pi}{\omega}\right)\frac{1}{r}\eta'(r)\right] r^{\pm\pi/\omega} \sin(\frac{\pi}{\omega}\theta)$$

we have

Calculation of Coefficient κ

Singular part $(**)$:

$$\int_\Omega (u_S \Delta s_- - s_- \Delta u_S)\,\mathrm{d}x = \kappa \int_\Omega (s_+ \Delta s_- - s_- \Delta s_+)\,\mathrm{d}x$$

Since

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we have

$$\begin{aligned}
&\int_\Omega (s_+ \Delta s_- - s_- \Delta s_+)\,\mathrm{d}x \\
&\qquad = -\frac{4\pi}{\omega}\int_\Omega \left(\frac{1}{r}\sin^2(\frac{\pi}{\omega}\theta)\eta\eta'\right)\mathrm{d}x \\
&\qquad = -\frac{4\pi}{\omega}\underbrace{\left(\int_0^\omega \mathrm{d}\theta\right)}_{\omega/2}\underbrace{\left(\int_0^\infty \eta(r)\eta'(r)\mathrm{d}r\right)}_{-1/2} \\
&\qquad = \pi.
\end{aligned}$$

Calculation of Coefficient κ

Combining $(*)$ and $(**)$, we obtain

$$\begin{aligned}
&\int_\Omega (u\Delta s_- - s_-\Delta u)\,\mathrm{d}x \\
&= \underbrace{\int_\Omega (w\Delta s_- - s_-\Delta w)\,\mathrm{d}x}_{(*)} + \underbrace{\int_\Omega (u_S\Delta s_- - s_-\Delta u_S)\,\mathrm{d}x}_{(**)} \\
&= 0 + \kappa\pi,
\end{aligned}$$

and thus

$$\begin{aligned}
\kappa &= \frac{1}{\pi}\int_\Omega (u\Delta s_- - s_-\Delta u)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_\Omega (u\Delta s_- + s_- f)\,\mathrm{d}x
\end{aligned}$$

Calculation of Coefficient κ

Alternative respresentation:

Using

$$\int_\Omega u\Delta s_- = -\int_\Omega \nabla R\left[(-\Delta)s_-\right]\cdot\nabla u dx = \int_\Omega R\left[(-\Delta)s_-\right]\Delta u dx$$

we have

Calculation of Coefficient κ

Alternative respresentation:

Using

$$\int_\Omega u \Delta s_- = -\int_\Omega \nabla R\left[(-\Delta)s_-\right] \cdot \nabla u dx = \int_\Omega R\left[(-\Delta)s_-\right] \Delta u dx$$

we have

$$\begin{aligned} \kappa &= \frac{1}{\pi}\int_\Omega (u\Delta s_- - s_-\Delta u)\, dx \\ &= \frac{1}{\pi}\int_\Omega \left(s_- - R\left[(-\Delta)s_-\right]\right) \underbrace{(-\Delta)u}_{f} dx \end{aligned}$$

Calculation of Coefficient κ

Alternative respresentation:

Using

$$\int_\Omega u\Delta s_- = -\int_\Omega \nabla R\left[(-\Delta)s_-\right]\cdot\nabla u\mathrm{d}x = \int_\Omega R\left[(-\Delta)s_-\right]\Delta u\mathrm{d}x$$

we have

$$\begin{aligned}\kappa &= \frac{1}{\pi}\int_\Omega (u\Delta s_- - s_-\Delta u)\,\mathrm{d}x\\ &= \frac{1}{\pi}\int_\Omega \left(s_- - R\left[(-\Delta)s_-\right]\right)\underbrace{(-\Delta)u}_{f}\mathrm{d}x\end{aligned}$$

Summary: Stress intensity factor

$$\begin{aligned}\kappa &= \frac{1}{\pi}\int_\Omega (u\Delta s_- + s_- f)\,\mathrm{d}x\\ &= \frac{1}{\pi}\int_\Omega \left(s_- - R\left[(-\Delta)s_-\right]\right) f\ \mathrm{d}x\end{aligned}$$

Outline

Standard FEM Estimates

We know: $$u \in H^r(\Omega) \quad \forall r < 1 + \frac{\pi}{\omega}$$

Let $\mathcal{T}_h$ be a quasi-uniform grid, $V_h \subset H_0^1(\Omega)$ piecewise linear FE space, $\tilde{u}_h \in V_h$ satisfies

$$\int_\Omega \nabla \tilde{u}_h \cdot \nabla v \, \mathrm{d}x = \int_\Omega f \, v \, \mathrm{d}x \quad \forall v \in V_h.$$

Then:

$$|u - \tilde{u}_h|_{H^1(\Omega)} \lesssim_\epsilon h^{(\pi/\omega)-\epsilon} \|f\|_{L^2(\Omega)},$$
$$\|u - \tilde{u}_h\|_{L^2(\Omega)} \lesssim_\epsilon h^{2(\pi/\omega)-\epsilon} \|f\|_{L^2(\Omega)},$$

Standard FEM Estimates

We know:

$$u \in H^r(\Omega) \quad \forall r < 1 + \frac{\pi}{\omega}$$

$$\kappa = \frac{1}{\pi} \int_\Omega (u \Delta s_- + s_- f)\, \mathrm{d}x$$

Let $\mathcal{T}_h$ be a quasi-uniform grid, $V_h \subset H_0^1(\Omega)$ piecewise linear FE space, $\tilde{u}_h \in V_h$ satisfies

$$\int_\Omega \nabla \tilde{u}_h \cdot \nabla v \, \mathrm{d}x = \int_\Omega f\, v \, \mathrm{d}x \quad \forall v \in V_h.$$

Let

$$\tilde{\kappa}_h = \frac{1}{\pi} \int_\Omega (\tilde{u}_h \Delta s_- + s_- f)\, \mathrm{d}x$$

Then:

$$|u - \tilde{u}_h|_{H^1(\Omega)} \lesssim_\epsilon h^{(\pi/\omega)-\epsilon} \|f\|_{L^2(\Omega)},$$

$$\|u - \tilde{u}_h\|_{L^2(\Omega)} \lesssim_\epsilon h^{2(\pi/\omega)-\epsilon} \|f\|_{L^2(\Omega)},$$

$$|\kappa - \tilde{\kappa}_h| \lesssim_\epsilon h^{2(\pi/\omega)-\epsilon} \|f\|_{L^2(\Omega)}$$

Multigrid Method I

Idea: $u = w + \kappa s_+$
Let $f \in L^2(\Omega)$. Consider

$$\begin{aligned} -\Delta w &= f + \kappa \Delta s_+ && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega \end{aligned}$$

- If κ was known: good convergence rate (since $w \in H^2(\Omega)$)
- Unfortunately κ unknown

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Idea: $u = w + \kappa s_+$

Let $f \in L^2(\Omega)$. Consider

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- If κ was known: good convergence rate (since $w \in H^2(\Omega)$)
- Unfortunately κ unknown

- Use solution u_{k-1} on coarser mesh to approximate κ:

$$\kappa \approx \kappa_k = \frac{1}{\pi} \int_\Omega u_{k-1} \Delta s_- + s_- f \, dx$$

- Solve

$$\begin{aligned} -\Delta w_k &= f + \kappa_k \Delta s_+ && \text{in } \Omega \\ w_K &= 0 && \text{on } \partial\Omega \end{aligned}$$

  for w_k

- $u_k = w_k + \kappa_k s_+$

Multigrid Method II

Consider $f \in H^1(\Omega)$. Then the solution u of the Poisson-Dirichlet problem on an Γ-shaped domain Ω can be written as

$$u = w + \sum_{\ell \in \mathcal{L}} \kappa_\ell s_{+,\ell}$$

where $w \in H^{3-\epsilon}(\Omega) \cap H_0^1(\Omega)$ for any $\epsilon > 0$ where

$$\mathcal{L} = \{\ell \in \mathbb{N} : \ell \frac{\pi}{\omega} < 2\} \text{ and}$$

$$s_{+,\ell}(r,\theta) = \eta(r) r^{\ell\pi/\omega} \sin(\ell \frac{\pi}{\omega} \theta)$$

Multigrid Algorithm II:

- similar idea as before
- produces approximate solution

$$u_k = \sum_{\ell \in \mathcal{L}} \kappa_{\ell,k} s_{+,\ell} + w_k$$

where w_k piecewise linear function on level k.

Multigrid Method II

End of part I

Thank you for your attention!