Vertex Behavior in 2D

Seminar "Elliptic Problems on Non-Smooth Domains"

JKU Linz WS 2013/14

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November 26, 2013

What happened so far...

For given $f \in L^2(\Omega)$, we are interested in regularity of u solving

$$\Delta u = f$$
$$+ \text{ BCs}$$

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- Variational solution:

$$\Delta : V \subset H^1(\Omega) \to L^2(\Omega)$$

is isomorphism, i.e., $f \in L^2(\Omega) \Longrightarrow u \in H^1(\Omega)$

We want more!

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  - Corner behavior ?

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- **Goal:** Augment space V^2 by space $X \subset H^s(\Omega)$, $s \in (1, 2)$ s.t. Δ surjective onto $L^2(\Omega)$

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Procedure:
1) get $dim(N) < \infty$ (Fredholm property)
2) define X s.t. Δ surjective onto $L^2(\Omega)$

Outline

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1. Fredholm Property

2. Derivation of Singular Solutions

Fredholm Property

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- Characterization of N (see talk by W. Krendl):

Dirichlet BVP

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$$v \in N \quad \Longleftrightarrow$$

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Neumann BVP

$$v \in N \qquad \Longleftrightarrow \qquad \begin{array}{l} \Delta v = 0 \\ \\ \gamma_j\left(\frac{\partial v}{\partial \nu_j}\right) = 0,\ j \in \mathcal{N} \\ \int_\Omega v \Delta \eta_j dx = 0 \text{ for } j \in \mathcal{N}^2 \end{array}$$

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- $v \in N \implies v \in C^\infty\left(\overline{\Omega} \setminus \cup_j U_\delta(S_j)\right)$

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Investigate $v \in N$ near Dirichlet corner S_j

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Investigate $v \in N$ near Dirichlet corner S_j

- Transition to polar coordinates with origin S_j

$$\begin{aligned} 0 &= \Delta_{(x,y)} v \\ &= \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}, \qquad 0 < \theta < \omega_j, 0 < r < \rho \end{aligned}$$

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Investigate $v \in N$ near Dirichlet corner S_j

Proposition (Expansion of $v \in N$ near *Dirichlet* corner S_j)

Let $v \in N$. Then

$$v(r,\theta) = \sum_{m\geq 1} \alpha_m r^{\lambda_{j,m}} \varphi_{j,m}(\theta) + \sum_{0<\lambda_{j,m}<1} \beta_m r^{-\lambda_{j,m}} \varphi_{j,m}(\theta)$$

for all $(r,\theta) \in (0,\rho) \times (0,\omega_j)$ where α_m, $\beta_m \in \mathbb{R}$ and

$$|\alpha_m| \leq L\sqrt{m}\,\rho^{-\lambda_{j,m}}$$

where L is a constant depending only on v.

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Proof: *Blackboard*

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Neumann Corner S_j:

Ansatz: $v(r,\theta) = r^\lambda \varphi(\theta)$ where $\varphi \in \{\varphi \in H^2(0,\omega_j) : \varphi'(0) = \varphi'(\omega_j) = 0\}$

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$$\implies \qquad \lambda_{j,1} = 0 \qquad\qquad \varphi_{j,1}(\theta) = \sqrt{\frac{1}{\omega_j}}$$

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Mixed Corner S_j ($j \in \mathcal{N}$, $j+1 \in \mathcal{D}$):
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Match expansions together

Lemma

For each j and each $\lambda_{j,m} \in (0,1)$ there exists $\sigma_{j,m} \in N$ such that

$$\sigma_{j,m} \; - \eta_j \, r_j^{-\lambda_{j,m}} \, \varphi_{j,m}(\theta) \; \in H^1(\Omega).$$

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Theorem

$$dim(N) = \sum_j \mathrm{card}\{\lambda_{j,m} : 0 < \lambda_{j,m} < 1\}$$

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$$\begin{cases} 0 \text{ if } \omega_j \leq \pi/2 \\ 1 \text{ if } \pi/2 < \omega_j \leq 3\pi/2 \\ 2 \text{ if } \omega_j > 3\pi/2 \end{cases}$$

Outline

Derivation of Singular Solutions

$$\Delta : V^2 + X \to \mathcal{R}(\Delta; V^2) + N = L^2(\Omega)$$

Definition of X:

- $\Delta(V^2 + X) = L^2(\Omega)$
- $X \subset H^s(\Omega)$, $1 < s < 2$

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Lemma

$S_{j,m}$ is the variational solution to

$$\Delta S_{j,m} = F_{j,m}$$
$$\gamma_k S_{j,m} = 0 \text{ for } k \in \mathcal{D}$$
$$\gamma_k \frac{\partial S_{j,m}}{\partial \nu_k} = 0 \text{ for } k \in \mathcal{N}.$$

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Theorem

Let Ω be a bounded, polygonal, open subset of $\mathbb{R}^2$. For each $f \in L^2(\Omega)$ there exists a unique $u \in V$ (variational) solution of

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega f \, v \, dx$$

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Corollary

- For each corner S_j it holds: $u \in H^s(U_\delta(S_j))$ for every $s \leq 2$ such that $s < 1 + \inf_m \{\lambda_{j,m} : 0 < \lambda_{j,m} < 1\}$.

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Remark

Conclusion remains valid in case of non-homogeneous BCs:
Let $f \in L^2(\Omega)$, $g_j \in H^{3/2}(\Gamma_j)$ for $j \in \mathcal{D}$, $h_j \in H^{1/2}(\Gamma_j)$ for $j \in \mathcal{N}$ satisfying the compatibility conditions

- $g_j(S_j) = g_{j+1}(S_j)$ for $j \in \mathcal{D}$
- $h_j \equiv \frac{\partial g_{j+1}}{\partial \nu_j}$ at S_j for $j \in \mathcal{M}'$ (i.e. $j+1 \in \mathcal{D}$, $j \in \mathcal{N}$, $\omega_j = \frac{\pi}{2} or \frac{3\pi}{2}$)
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Then there exists a unique u of the form
$u = u_R + \sum_j \left(\sum_{0<\lambda_{j,m}<1} c_{j,m} S_{j,m} \right)$ with $u_R \in H^2(\Omega)$ solution of

$$\Delta u = f \text{ in } \Omega$$
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Dirichlet problem: Let $\Lambda = \pi / max_j \omega_j$ $(> 1/2)$.

- Assume $u \in H_0^1(\Omega)$ and $\Delta u \in L^2(\Omega)$. Then $u \in H^s(\Omega)$ for all $s \leq 2$ with $s < 1 + \Lambda$.

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Thus:

$u \in H^t(\Omega)$ with $t \geq 0$ and $\geq 1 - \Lambda$, solution of $\Delta u = f \in L^2(\Omega)$ with $\gamma_j u = 0$ for every j

$$\Longrightarrow$$

$$u \in H^s(\Omega) \text{ for all } s \leq 2 \text{ with } s < 1 + \Lambda$$