Solution Technique in the Neighborhood of Singular Points

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1. Condition Number Estimate

2. Discretization Error Estimates
   - Discretization Error Estimates in $\Omega \setminus K_\delta$
   - Discretization Error Estimates in K_δ

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Condition Number Estimate

Model Problem

$$
\begin{aligned}
-\Delta u + u &= f && \text{in } \Omega \\
\frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Condition Number

$$
cond_2(L) = \frac{\lambda_{max}(L)}{\lambda_{min}(L)} \leq c\frac{1}{h^2}
$$

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Discretization Error Estimates

Model Problem

$$
\begin{aligned}
-div(A\nabla u) + bu &= f && in\ \Omega \\
u &= 0 && on\ \Gamma_D \\
\frac{\partial u}{\partial n} + \sigma u &= 0 && on\ \Gamma_R
\end{aligned}
$$

- the origin is the only singular point
- linear FE with discretization parameter h

$$\|u - \widetilde{v}\|_{L_2(\Omega)} \leq ch^{2\lambda} \|f\|_{L_2(\Omega)} \tag{1}$$

with $\lambda = \frac{\pi}{\beta}$.

Discretization Error Estimates

Desired Error Estimate

$$\|u - \widetilde{v}\|_{W_2^1(\Omega \setminus K_\delta)} \leq ch \|f\|_{L_2(\Omega)}$$

$$\left\|u - \widetilde{\widetilde{v}}\right\|_{W_2^1(K_\delta)} \leq ch \|f\|_{L_2(\Omega)}$$

thus:

$$\|u - v_1\|_{W_2^1(\Omega)} \leq ch \|f\|_{L_2(\Omega)}$$

Discretization Error Estimates in $\Omega \setminus K_\delta$

Construct Cut-Off Function Q

1. $Q \in W^2_\infty(\Omega)$
2. $Q(x,y) = \begin{cases} 1 & : (x,y) \notin K_\delta \\ 0 & : (x,y) \in K_{\kappa\delta}, 0 < \kappa < 1 \end{cases}$
3. $\left.\frac{\partial Q}{\partial N}\right|_{S=\partial\Omega} = 0, \quad \text{with } \frac{\partial Q}{\partial N} := a_{ij} \frac{\partial Q}{\partial x_j} \cos(n, x_i)$

Discretization Error Estimates in $\Omega \setminus K_\delta$

$$q(x,y) := \left(x - y\frac{a_{12}(x,0)}{a_{22}(x,0)} \right)^2 + Ky^2$$

$$q_1(x,y) := \xi\left(\frac{q(x,y)}{c_1^2\delta^2} \right)$$

$$q_2(x,y) := \xi\left(\frac{q(x,y)}{c_2^2\delta^2} \right)$$

we define

$$Q(x,y) := \begin{cases} q_1(x,y)\cos^2\left(\frac{\pi\theta}{2\beta}\right) + q_2(x,y)\sin^2\left(\frac{\pi\theta}{2\beta}\right) & : (x,y) \in \Omega \cap K_\delta \\ 1 & : (x,y) \in \Omega \setminus K_\delta \end{cases}$$

Discretization Error Estimates in $\Omega \setminus K_\delta$

$$L_\Omega(\widetilde{v} - \widetilde{u}, \phi) = L_\Omega(u - \widetilde{u}, \phi); \quad \forall \phi \in H_h$$

defining $\widetilde{w} := \widetilde{v} - \widetilde{u}$ and $\phi := \widetilde{Q\widetilde{w}}$ we get:

$$L_\Omega(\widetilde{w}, Q\widetilde{w}) =$$
$$L_\Omega(\widetilde{w}, Q\widetilde{w} - \widetilde{Q\widetilde{w}}) + L_\Omega(u - \widetilde{u}, \widetilde{Q\widetilde{w}} - Q\widetilde{w}) + L_\Omega(u - \widetilde{u}, Q\widetilde{w})$$

further:

$$L_\Omega(\widetilde{w}, Q\widetilde{w}) = \int_\Omega Q\left(a_{ij}\frac{\partial \widetilde{w}}{\partial x_i}\frac{\partial \widetilde{w}}{\partial x_j} + b\widetilde{w}^2\right) d\Omega + \int_S \sigma Q \widetilde{w}^2 ds$$
$$+ \int_\Omega \widetilde{w} a_{ij} \frac{\partial Q}{\partial x_i}\frac{\partial \widetilde{w}}{\partial x_j} d\Omega$$

Discretization Error Estimates in $\Omega \setminus K_\delta$

partial integration yields:

$$\int_\Omega Q \left(a_{ij} \frac{\partial \widetilde{w}}{\partial x_i} \frac{\partial \widetilde{w}}{\partial x_j} + b\widetilde{w}^2 \right) d\Omega + \int_S \sigma Q \widetilde{w}^2 ds$$
$$= L_\Omega(\widetilde{w}, Q\widetilde{w} - \widetilde{Q\widetilde{w}}) + L_\Omega(u - \widetilde{u}, \widetilde{Q\widetilde{w}} - Q\widetilde{w})$$
$$+ L_\Omega(u - \widetilde{u}, Q\widetilde{w}) + \frac{1}{2} \int_\Omega \widetilde{w}^2 \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial Q}{\partial x_i} \right) d\Omega$$

Discretization Error Estimates in $\Omega \setminus K_\delta$

After estimation of all the terms we arrive at:

$$(1 - C\epsilon) \int_\Omega Q a_{ij} \left(\left| \nabla \widetilde{w} \right|^2 + b \widetilde{w}^2 \right) d\Omega + \int_S \sigma Q \widetilde{w}^2 ds$$

$$\leq ch \| \widetilde{w} \|^2_{W_2^1(\Omega)} + ch^2 (\| f \|_{L_2(\Omega)} \| \widetilde{w} \|_{W_2^1(\Omega)} + \| f \|^2_{L_2(\Omega)})$$

$$+ ch^2 \| f \|_{L_2(\Omega)} \| \widetilde{w} \|_{L_2(\Omega)} + c \| \widetilde{w} \|^2_{L_2(\Omega)}$$

$$\leq ch^2 \| f \|_{L_2(\Omega)}$$

thus

$$\| u - \widetilde{v} \|_{W_2^1(\Omega \setminus K_\delta)} \leq ch \, \| f \|_{L_2(\Omega)}$$

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Discretization Error Estimates in K_δ

Denote by $\widetilde{\widetilde{v}}$ the solution of the BVP on the graded mesh:

$$L(\widetilde{\widetilde{v}}, \varphi) = \int_d \left(a_{ij} \frac{\partial \widetilde{\widetilde{v}}}{\partial x_i} \frac{\partial \widetilde{\varphi}}{\partial x_j} + b \widetilde{\widetilde{v}} \right) d\Omega + \int_\gamma \sigma \widetilde{\widetilde{v}} \widetilde{\varphi} ds = (f, \widetilde{\varphi})_d$$

$$\forall \widetilde{\varphi} \in H_\hbar^0(d_{ex}^h)$$

Discretization Error Estimates in K_δ

With $w := u - u_0$ and $\widetilde{\widetilde{z}} = \widetilde{\widetilde{v}} - \widetilde{v}_0$ we get:

$$L(\widetilde{\widetilde{z}} - \widetilde{w}, \widetilde{\phi}) = L(v_0 - \widetilde{v}_0, \widetilde{\phi}) + L(u_0 - v_0, \widetilde{\phi}) + L(w - \widetilde{w}, \widetilde{\phi})$$

Note:

$\widetilde{v}_0, \widetilde{w}, \widetilde{\widetilde{z}}, \widetilde{\widetilde{v}}$ and $\widetilde{\phi}$	continuous, piecewise linear f.e. functions on the new mesh;
$\widetilde{v}$	continuous, piecewise linear functions on the original mesh

Discretization Error Estimates in K_δ

Setting $\widetilde{\phi} = \widetilde{\widetilde{z}} - \widetilde{w}$

$$\|\widetilde{\widetilde{z}} - \widetilde{w}\|_{W_2^1(d)} \leq \left(\|v_0 - \widetilde{v}_0\|_{W_2^1(d)} + \|u_0 - v_0\|_{W_2^1(d)} + \|w - \widetilde{w}\|_{W_2^1(d)}\right)$$

Discretization Error Estimates in K_δ

Estimating all terms:

$$\|\tilde{\tilde{z}} - \tilde{w}\|_{W_2^1(d)} \leq ch\|f\|_{L_2(\Omega)}$$

and we get

$$\|u - \tilde{\tilde{v}}\|_{W_2^1(d)} \leq ch\|f\|_{L_2(\Omega)}$$

Thank you for your attention!