Vertex behaviour in 3d

Armin Fohler

10^{th} December, 2013

Maximal regularity

..., instead of describing explicitly the part of the solution that does not belong to H^2, we look for the best exponent s such that the solution belongs to H^s.

For Ω bounded, polyhedral subset of $\mathbb{R}^3, f \in L^2(\Omega)$ find $u \in V$:

$$\int_\Omega \nabla u \nabla v dx = -\int_\Omega f v dx \qquad \forall v \in V \tag{2.1.1}$$

We know that $\exists! u \in V = \left\{ v \in H^1(\Omega) : \gamma_j v = 0 \text{ on } \Gamma_j ; j \in \mathcal{D} \right\}$

I) Faces

II) Edges

III) Vertices

For Ω bounded, polyhedral subset of $\mathbb{R}^3, f \in L^2(\Omega)$ find $u \in V$:

$$\int_\Omega \nabla u \nabla v dx = -\int_\Omega f v dx \qquad \forall v \in V \tag{2.1.1}$$

We know that $\exists! u \in V = \{v \in H^1(\Omega) : \gamma_j v = 0 \text{ on } \Gamma_j; j \in \mathcal{D}\}$

I) Faces

Theorem 2.1.4

$\varphi u \in H^2(\Omega)$ for every $\varphi \in D\left(\overline{\Omega}\right)$ whose support is part of the interior of Γ_j .

II) Edges

III) Vertices

For Ω bounded, polyhedral subset of $\mathbb{R}^3, f \in L^2(\Omega)$ find $u \in V$:

$$\int_\Omega \nabla u \nabla v dx = -\int_\Omega f v dx \qquad \forall v \in V \tag{2.1.1}$$

We know that $\exists! u \in V = \left\{ v \in H^1(\Omega) : \gamma_j v = 0 \text{ on } \Gamma_j ; j \in \mathcal{D} \right\}$

I) Faces

II) Edges

Proposition 2.6.1

$\varphi u \in H^s(\Omega)$ for every $s \leq 2$ with $s < \Lambda + 1$ and every $\varphi \in D\left(\overline{\Omega}\right)$ whose support is away from the vertices.

III) Vertices

For Ω bounded, polyhedral subset of $\mathbb{R}^3, f \in L^2(\Omega)$ find $u \in V$:

$$\int_\Omega \nabla u \nabla v dx = -\int_\Omega f v dx \qquad \forall v \in V \tag{2.1.1}$$

We know that $\exists! u \in V = \left\{ v \in H^1(\Omega) : \gamma_j v = 0 \text{ on } \Gamma_j ; j \in \mathcal{D} \right\}$

I) Faces

II) Edges

III) Vertices

Theorem 2.6.3

There exists unique numbers c_k such that

$$u - \sum_k c_k \rho^{-1/2+\sqrt{(\lambda_k+1/4)}} \psi_k(\sigma) \in H^s(V)$$

for every $s \leq 2$ with $s < \Lambda + 1$; and $\lambda_k \geq s^2 - 2s + \frac{3}{4}$.

The Main Theorem

Theorem 2.6.3

Let Ω be a bounded polyhedral open subset of $\mathbb{R}^3$. For $f \in L^2(\Omega)$ let u be the solution of (2.1.1) then there exists unique numbers c_k such that

$$u - \sum_k c_k \rho^{-1/2+\sqrt{(\lambda_k+1/4)}} \psi_k(\sigma) \in H^s(V)$$

for every $s \leq 2$ with $s < \Lambda + 1$, where the sum is over the k such that $\lambda_k \leq s^2 - 2s + \frac{3}{4}$.

$$\int_\Omega \nabla u \nabla v dx = - \int_\Omega f v dx \tag{2.1.1}$$

Transformation to spherical coordinates yields:

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\Delta' u = f \tag{2.6.1}$$

with the Laplace-Beltrami-Operator $\Delta' u$ on S^2

$$v \mapsto \frac{1}{sin\varphi}\frac{sin\varphi\, \partial v / \partial \varphi}{\partial \varphi} + (sin^2\varphi)^{-2}\frac{\partial^2 v}{\partial \theta^2}$$

We define the Operator $\mathcal{B}$

$$b(v,w) = \int_G \left(sin\varphi \frac{\partial v}{\partial \varphi}\frac{\partial w}{\partial \varphi} + \frac{1}{sin\varphi}\frac{\partial v}{\partial \theta}\frac{\partial w}{\partial \theta} \right) d\varphi d\theta$$

for v, w $\in \mathcal{V}$.

$$\mathcal{B}v = -\Delta' v \qquad \text{for } v \in D_{\mathcal{B}};$$

$$D_{\mathcal{B}} = \left\{ v \in \mathcal{V}; \Delta' v \in L^2(G) \text{ and } b(v,w) = -\int_G \Delta' v w d\sigma \quad \forall w \in \mathcal{V} \right\}$$

$\mathcal{B}$ is a self-adjoint operator in $\mathcal{H}$

with eigenvalues $\lambda_k \in \mathbb{R}_0^+$ for $k = 1, 2, ...$
and eigenfunctions $\psi_k \in D_{\mathcal{B}}$. Thus

$$\mathcal{B}\psi_k = \lambda_k \psi_k \text{ in } G.$$

With $\rho = e^t$ $(\rho < R)$ and $v(t,\sigma) = e^{(-s+3/2)t}u(e^t\sigma)$;

$$g(t,\sigma) = e^{(-s+7/2)t}f(e^t\sigma);$$

(2.6.1) becomes:

$$\frac{\partial^2 v}{\partial t^2} + 2(s-1)\frac{\partial v}{\partial t} + \Delta' v + (s-\frac{1}{2})(s-\frac{3}{2})v = g \quad (2.6.3)$$

in $(-\infty, lnR) \times G$ and $v(t,.) \in D_{\mathcal{B}} \quad \forall t$.

Lemma 2.6.4

Assume $(s - \frac{1}{2})(s - \frac{3}{2})$ is not an eigenvalue of $-\mathcal{B}$.
Then there exists $v_0 \in H^s(\mathbb{R} \times G)$ solution of (2.6.3) in $\mathbb{R} \times G$ and such that $v_0(t, .) \in D_{\mathcal{B}} \quad \forall t \in \mathbb{R}$.

using:

Lemma 2.6.2

One has $D_{\mathcal{B}} \subset H^s(G)$ for every $s \leq 2$ such that $s < \Lambda + 1$.

Therefore

$$v_0 \in H^s(\mathbb{R} \times G);$$

Using an inverse transformation we define u_0 on C:

$$u_0(\rho\sigma) = \rho^{s-3/2} v_0(ln\rho, \sigma). \qquad (2)$$

with

Lemma 2.6.5

Assume $\varphi \in H^s((-\infty, lnR) \times G)$ then $\rho^{s-3/2}\varphi \in H^s(C(R))$ $\forall s \geq 0$.

we get $u_0 \in H^s(C(R))$.

The following holds:

$$\Delta u_0 = f$$

and u_0 fulfills the same boundary conditions as u, since $v_0(ln\rho, .) \in D_{\mathcal{B}} \quad \forall \rho$.

Thus $u - u_0 \in D_{\mathcal{B}}$ and we get that $(u - u_0) \in H^1(C(R))$.

Inserting in (2.6.3) yields

$$\frac{\partial^2 (u - u_0)}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial (u - u_0)}{\partial \rho} + \frac{1}{\rho^2} \Delta' (u - u_0) = 0$$

expressed in eigenfunctions of $\mathcal{B}$ we write:

$$u - u_0 = \sum_{k \geq 1} (a_k \rho^{\alpha_k} + b_k \rho^{\beta_k}) \psi_k(\sigma)$$

Since

Theorem 1.2.19

... let u be a function which is smooth in $\overline{\Omega} \setminus 0$ and which coincides with $\rho^\alpha \varphi(\sigma)$ in $V \cap \Omega$ where $\varphi \in H^{s_0}(G)$. Then for every $s < s_0$ one has

$$u \in H^s(\Omega) \text{ for } Re(\alpha) > s - 1$$
$$u \notin H^s(\Omega) \text{ for } Re(\alpha) \leq s - 1$$

when $Re(\alpha)$ is not an integer.

we get $b_k = 0$.

The rest corresponding to $\alpha_k > s - 3/2$ converges in $H^s(C(R'))$ for $R' < R$, because of

Lemma 2.6.6

The functions $\rho^{\alpha_k}\psi_k(\sigma)$ belong to $H^s(C(R))$ for $\alpha_k > s - 3/2$ and in addition $\|\rho^{\alpha_k}\psi_k(\sigma)\|_{s,C(R)} = O(kR^{\alpha_k})$.

with Parseval's identity

$$\sum_{k\geq 1} a_k^2 R^{2\alpha_k} = \int_G |(u-u_0)(R\sigma)|^2 d\sigma < \infty$$

we get:

$$\|\sum_k a_k \rho^{\alpha_k}\psi_k\|_{s,C(R')} \leq O(\sum_k kR'^{\alpha_k}|a_k|) < \infty$$

for $\alpha_k > s - 3/2$ or equivalently: $\lambda_k \geq s^2 - 2s + 3/4$.

This concludes the proof of Theorem 2.6.3.

Corollary 2.6.7

Let Ω be any bounded polyhedral open subset of $\mathbb{R}^3$, then there exists $s_0 \geq \frac{3}{2}$ such that for every $f \in L^2(\Omega)$ the variational solution u of the problem (2.1.1), in the case of pure Dirichlet or pure Neumann boundary condition, belongs to $H^s(\Omega)$ for every $s < s_0$.

Corollary 2.6.8

Let Ω be a convex bounded polyhedral open subset of $\mathbb{R}^3$, then for every $f \in L^2(\Omega)$ the variational solution u of the problem (2.1.1), in the case of pure Dirichlet boundary condition, belongs to $H^2(\Omega)$.

Corollary 2.6.9

A similar H^2 regularity result in any convex polyhedron for a pure Neumann boundary value problem is also true. However the above method (relying on a monotonicity property of the eigenvalues of the Laplace-Beltrami operator) does not work.

Theorem 2.5.12

Under the assumptions of Theorem 2.5.11 ($f \in L^2(Q)$; u solution of $\int_Q \nabla u \nabla v dx = -\int_Q f v dx$) and given j, let V_j be an open neighborhood of S_j in $\overline{\Omega}$ which does not contain any other corner, then u belongs to $H^s(V_j \times \mathbb{R})$ for every $s \leq 2$ such that $s < \lambda_{j,m} + 1$ for all $\lambda_{j,m}$ such that $0 < \lambda_{j,m} < 1$.