Singular basis functions

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Motivating example

One-dimensional BVP:

$$-(ay')' = f, \quad x \in (0,1),$$
$$y(0) = y(1) = 0$$

where

- $a \geq a_0 > 0$ with discontinuity at $x = \xi \quad (0 < \xi < 1)$
- right-hand side f is piecewise smooth.

Motivating example

Thence we reformulate the problem: state equation

$$-(ay')' = f, \quad 0 < x < \xi \text{ and } \xi < x < 1, \qquad (1)$$
$$y(0) = y(1) = 0$$

and continuity conditions

$$\lim_{x \to \xi - 0} y(x) = \lim_{x \to \xi + 0} y(x),$$
$$\lim_{x \to \xi - 0} ay' = \lim_{x \to \xi + 0} ay',$$

or, briefly,

$$[y]_{x=\xi} = 0, \quad [ay']_{x=\xi} = 0. \qquad (2)$$

Motivating example

Even for smooth right-hand side f, usually the solution $y \notin H^2((0,1))$.

Example

Let $\xi = 1/2$, $f \equiv 1$, $a = \begin{cases} 2, & 0 \leq x \leq 1/2 \\ 1, & 1/2 < x \leq 1 \end{cases}$.

Then the solution is

$$y = \begin{cases} -\frac{x^2}{4} + \frac{7}{24}x, & 0 \leq x \leq 1/2 \\ -\frac{x^2}{2} + \frac{7}{12}x - \frac{1}{12}, & 1/2 < x \leq 1 \end{cases}.$$

One can see $y \in H^1((0,1))$, but $y \notin H^2((0,1))$
(since y' is discontinuous).

Motivating example

Multiply the equation

$$-(ay')' = f, \quad 0 < x < \xi \text{ and } \xi < x < 1,$$

by an arbitrary function $\phi \in H_0^1((0,1))$ and integrate on $(0,1)$. Then using integration by parts on $(0,\xi)$ and $(\xi,1)$ and jump condition from (2), we deduce

$$\int_0^1 ay'\phi' \, dx = \int_0^1 f\phi \, dx. \tag{3}$$

The converse statement is also true: if $y \in H_0^1((0,1))$ satisfies (3) for any $\phi \in H_0^1((0,1))$, then it satisfies the system (1) and the conditions (2).

Variational numerical scheme in 1D

We focus on the variational solution y of the problem (3). Consider a variational scheme based on piecewise linear approximations and corresponding order of approximation.
Also consider on $(0,1)$ a regular mesh $x_i = ih$, $i = 1, \ldots, N$, $h = 1/N$.
Approximate solution $\tilde{v} \in V_h \subset H_0^1((0,1))$ satisfies the integral equation

$$\int_0^1 a\tilde{v}'\tilde{\phi}'\, dx = \int_0^1 f\tilde{\phi}\, dx$$

for any $\tilde{\phi} \in V_h$.
For the approximation error $(y - \tilde{v})$ we have an estimate

$$\|y - \tilde{v}\|_{1,(0,1)} \leq C\|y - \tilde{y}\|_{1,(0,1)},$$

where $\tilde{y}$ is a nodal interpolant of $y(x)$.

Variational numerical scheme in 1D

When the point of discontinuity ξ coincides with one meshnode, the terms of

$$\|y-\tilde{y}\|^2_{1,(0,1)} = \|y-\tilde{y}\|^2_{1,(0,\xi)} + \|y-\tilde{y}\|^2_{1,(\xi,1)}$$

are estimated above by $Ch^2\|y\|^2_{2,(0,\xi)}$ and $Ch^2\|y\|^2_{2,(\xi,1)}$.
As a result,

$$\|y-\tilde{v}\|_{1,(0,1)} = O(h).$$

Variational numerical scheme in 1D

When there is no meshnode in $O(h^2)$-neighborhood of singular point ξ:

- for simplicity $\xi = 1/2$
- N is odd
- closest meshnodes are $x_L = 1/2 - h/2$, $x_R = 1/2 + h/2$

Then

$$\|y - \tilde{v}\|_{1,(0,1)}^2 \geq \int_{x_L}^{x_R} (y' - \tilde{v}')^2 \, dx \geq \min_{\alpha} \int_{x_L}^{x_R} (y' - \alpha)^2 \, dx.$$

Discontinuity of y' at $1/2$ yields that

$$\min_{\alpha} \int_{x_L}^{x_R} (y' - \alpha)^2 \, dx \geq Ch + O(h^2)$$
$$\Rightarrow \ \|y - \tilde{v}\|_{1,(0,1)} \geq Ch^{1/2}.$$

2D problem with discontinuous coefficients

In the domain $\Omega \subset \mathbb{R}^2$ with smooth boundary S consider the equation

$$Lu \equiv -\frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + b_i \frac{\partial u}{\partial x_i} + au = f \tag{4}$$

with one type of boundary conditions

$$u\Big|_S = 0, \tag{5}$$

$$\left(\frac{\partial u}{\partial N} + \sigma u \right)\Bigg|_S = 0. \tag{6}$$

2D problem with discontinuous coefficients

Conditions on the data:

- coefficients b_1, b_2, a and right-hand side f are bounded piecewise smooth functons
- coefficients a_{ij} have discontinuities along smooth closed curve $\Gamma \subset \Omega$
- a_{ij} are bounded and continuous on Ω_1 (bounded by Γ) and $\Omega_2 = \Omega \setminus \overline{\Omega}_1$
- $\frac{\partial a_{ij}}{\partial x_k}$ are bounded and piecewise smooth on Ω_1 and Ω_2

2D problem with discontinuous coefficients

For the problem (4), (5) or (4), (6) we require on the curve Γ

$$[u]\big|_\Gamma = 0, \quad \left[\frac{\partial u}{\partial N}\right]\bigg|_\Gamma = 0, \tag{7}$$

where

$$\left[\frac{\partial u}{\partial N}\right]\bigg|_\Gamma = a_{ij}^+ \frac{\partial u^+}{\partial x_i}\cos(\nu, x_j) - a_{ij}^- \frac{\partial u^-}{\partial x_i}\cos(\nu, x_j).$$

Classical solution satisfies

- $u(x) \in C(\bar{\Omega}) \cap C^1(\Omega_i), \ i = 1, 2$
- $u(x) \in C^2(\Omega_i), \ i = 1, 2$
- $Lu = f$ on $\Omega_1 \cup \Omega_2$

2D problem with discontinuous coefficients

Define generalized solution for (4), (5), (7) with $f \in L^2(\Omega)$: $u \in H_0^1(\Omega)$ satisfies

$$L_\Omega(u,\varphi) = \int_\Omega \left[a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + b_i \frac{\partial u}{\partial x_i} \varphi + a u \varphi \right] \, d\Omega = \int_\Omega f \varphi \, d\Omega$$

for any $\varphi \in H_0^1(\Omega)$.

2D problem with discontinuous coefficients

Define generalized solution for (4), (5), (7) with $f \in L^2(\Omega)$: $u \in H_0^1(\Omega)$ satisfies

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for any $\varphi \in H_0^1(\Omega)$.

If $u \in H^2(\Omega_i)$, $i = 1, 2$, then (integrating by parts)

$$\int_\Omega (Lu - f)\varphi \, d\Omega + \int_\Gamma \varphi \left[\frac{\partial u}{\partial N} \right] ds = 0,$$

hence

$$Lu = f,$$
$$\left[\frac{\partial u}{\partial N} \right] \bigg|_\Gamma = 0.$$

2D problem with piecewise smooth boundary

Again consider in $\Omega \subset \mathbb{R}^2$ the equation

$$Lu \equiv -\frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + b_i \frac{\partial u}{\partial x_i} + au = f$$

with 1st, 3rd types of BC, or

$$u\Big|_{S_1} = 0, \quad \left(\frac{\partial u}{\partial N} + \sigma u\right)\Big|_{S_2} = 0, \tag{8}$$

where $S = S_1 \cup S_2$.
Coefficients and RHS of the equation satisfy regularity conditions (A).
But for mixed BC (8) in general $u \notin H^2(\Omega)$.

2D problem with piecewise smooth boundary

Generalized solution for mixed BC: $u \in H^1_{S_1}(\Omega)$ satisfies

$$L_{\Omega,S_2}(u,\varphi) \equiv \int_\Omega \left[a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j} + b_i \frac{\partial u}{\partial x_i}\varphi + au\varphi \right] d\Omega + \int_{S_2} \sigma u \varphi \, ds = (f,\varphi)_\Omega$$

for any $\varphi \in H^1_{S_1}(\Omega)$.

2D problem with piecewise smooth boundary

Consider ω – the sector of unit circle with angle β, and corresponding Dirichlet problem for Poisson equation:

$$-\triangle u = f, \quad u\big|_S = 0. \tag{9}$$

Function

$$\Psi = \zeta(r) r^{\lambda} \sin \lambda\theta$$

is a generalized solution of (9), where $\lambda = \pi/\beta$ and $\zeta = \begin{cases} 1, & 0 \le r \le 1/3 \\ 0, & 2/3 \le r \le 1 \end{cases}$ is monotone and smooth.

One can verify that $\Psi \in H_0^1(\omega)$.

2D problem with piecewise smooth boundary

Using

$$\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

one can show that $\triangle \Psi \in L^2(\omega)$.
Notice that

$$\|\Psi\|_{2,\omega}^2 \geq \int_\omega \left(\frac{\partial^2 \Psi}{\partial r^2}\right)^2 r\ dr\ d\theta \geq \int_0^{1/2}\int_0^\beta \lambda^2(\lambda-1)^2 r^{2\lambda-3}\sin^2\lambda\theta\ dr\ d\theta.$$

When $\pi < \beta < 2\pi$, we have $1/2 < \lambda < 1$ and $\Psi \notin H^2(\omega)$.

2D problem with piecewise smooth boundary

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Notice that

$$\|\Psi\|^2_{2,\omega} \geq \int_\omega \left(\frac{\partial^2 \Psi}{\partial r^2}\right)^2 r\ dr\ d\theta \geq \int_0^{1/2}\int_0^\beta \lambda^2(\lambda-1)^2 r^{2\lambda-3}\sin^2\lambda\theta\ dr\ d\theta.$$

When $\pi < \beta < 2\pi$, we have $1/2 < \lambda < 1$ and $\Psi \notin H^2(\omega)$.

Singular points on the boundary:

- corner points with inner angles $\pi < \beta < 2\pi$
- points of switching the boundary condition: $\overline{S}_1 \cap \overline{S}_2$

2D problem with piecewise smooth boundary

Theorem

Any solution of the stated problem with $f \in L^2(\Omega)$ can be expressed as

$$u = \sum_j \gamma_j \Psi_j + w,$$

where $w \in H^2(\Omega)$, γ_j are constant, $\Psi_j \in H^1(\Omega)$ are independent of f and

1. $L\Psi_j \in L^2(\Omega)$
2. *each singular point generates one or two functions Ψ_j; if BC is not switched, then exactly one function Ψ_j*
3. *Ψ_j is non-zero only near the corresponding singular point*
4. $\Psi_j\big|_{S_1} = 0$

In addition,

$$\sum_j |\gamma_j| + \|w\|_{2,\Omega} \leq C\|f\|_{0,\Omega}.$$

2D problem with piecewise smooth boundary

Exact representation of Ψ_j: find u

$$-\triangle u = 0 \text{ in } \omega$$
$$u(r,0) = u(r,\beta) = 0$$

in the form $u = r^\mu \Phi(\theta)$.
We have

$$\frac{d^2\Phi}{d\theta^2} + \mu^2\Phi = 0.$$

Since $\Phi(0) = \Phi(\beta) = 0$, non-trivial solution exists for $\mu_n = n\lambda$ ($\lambda = \pi/\beta$):

$$\Phi_n = \sin n\lambda\theta, \quad n = 1, 2, \ldots$$
$$\Rightarrow \; u_n = r^{n\lambda} \sin n\lambda\theta.$$

Since $u_n(r,\theta) \in H^2(\omega)$ $(n > 1)$, but $u_1(r,\theta) \notin H^2(\omega)$, a singular function has a form

$$\Psi = \zeta(r)u_1(r,\theta) = \zeta(r)r^\lambda \sin \lambda\theta.$$

2D problem with piecewise smooth boundary

Other cases of BC:

1. $\frac{\partial u}{\partial n}\Big|_{\theta=0} = \frac{\partial u}{\partial n}\Big|_{\theta=\beta} = 0,\ \pi < \beta < 2\pi$

$$\Rightarrow\ \Psi = \zeta(r) r^{\lambda} \cos \lambda\theta, \quad \lambda = \pi/\beta$$

2. $\frac{\partial u}{\partial n}\Big|_{\theta=0} = u\big|_{\theta=\beta} = 0,\ \pi/2 < \beta < 2\pi$

$$\Rightarrow\ \Psi_1 = \zeta(r) r^{\lambda_1} \cos \lambda_1\theta, \quad \lambda_1 = \pi/2\beta \quad \text{for } \pi/2 < \beta \leq 3\pi/2$$

and also

$$\Psi_2 = \zeta(r) r^{\lambda_2} \cos \lambda_2\theta, \quad \lambda_2 = 3\pi/2\beta \quad \text{for } 3\pi/2 < \beta < 2\pi$$

3. $u\big|_{\theta=0} = \frac{\partial u}{\partial n}\Big|_{\theta=\beta} = 0,\ \pi/2 < \beta < 2\pi$

$$\Rightarrow\ \Psi_1 = \zeta(r) r^{\lambda_1} \sin \lambda_1\theta, \quad \lambda_1 = \pi/2\beta \quad \text{for } \pi/2 < \beta \leq 3\pi/2$$

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2D problem with piecewise smooth boundary

Defining singular functions for general operator L with piecewise linear boundary around corner points:

1. change of variables

$$\eta_1 = x_1 + \mu x_2, \quad \eta_2 = \nu x_2$$

   to obtain $\tilde{L} = -\triangle_\eta + \tilde{b}_j \frac{\partial}{\partial \eta_j} + \tilde{a}$

2. for sufficiently small $\varepsilon > 0$ the example of singular function is (in polar coordinates (ρ, κ))

$$\Psi = \zeta(\rho/\varepsilon)\rho^\lambda \sin \lambda\kappa$$

Singularities for intersection of discontinuity curve with boundary

Arising singularity functions depend only on

- coefficients a_{ij} for 2nd derivatives at intersection points
- angles between discontinuity curve and jointed parts of the boundary

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Model problem:

$$\begin{cases} a^+ \triangle u^+ = f^+ \quad \text{in } \Omega_+ \\ a^- \triangle u^- = f^- \quad \text{in } \Omega_- \\ u\big|_S = 0 \\ u^+\big|_{x_2=0} = u^-\big|_{x_2=0}, \quad a^+ \frac{\partial u^+}{\partial n^+}\Big|_{x_2=0} = a^- \frac{\partial u^-}{\partial n^+}\Big|_{x_2=0} \end{cases}$$

where $u = \begin{cases} u^+ \text{ in } \Omega_+, \\ u^- \text{ in } \Omega_-, \end{cases} \quad f = \begin{cases} f^+ \text{ in } \Omega_+, \\ f^- \text{ in } \Omega_-, \end{cases}$, $\Omega_- = \Omega \setminus \overline{\Omega}_+$,
$\Omega_+ \cap \Omega_- \subset \{x \colon x_2 = 0\}$.

Singularities for intersection of discontinuity curve with boundary

Theorem

In the given problem for any $f \in L^2(\Omega)$ there exists a generalized solution $u \in H_0^1(\Omega)$ which can be written as

$$u = \sum_j \gamma_j \Psi_j + w,$$

where $\Psi_j \in H^1(\Omega)$ are independent of f, $w \in B^2(\Omega)$, $a^+ \triangle \Psi_j \in L^2(\Omega_+)$, $a^- \triangle \Psi_j \in L^2(\Omega_-)$. Number of Ψ_j is not greater than 2.

Singularities for intersection of discontinuity curve with boundary

Determining the functions Ψ_j:
solve the homogeneous problem in polar coordinates in the form
$u = r^{\mu}\Phi(\theta)$, where $\Phi(\theta) = \begin{cases} \Phi^+(\theta), & 0 < \theta < \beta_+, \\ \Phi^-(\theta), & -\beta_- < \theta < 0. \end{cases}$
Boundary conditions:

$$u^+(r, \beta_+) = u^-(r, -\beta_-) = 0.$$

The state equation reads as

$$a(\theta)\Phi''(\theta) + a(\theta)\mu^2\Phi(\theta) = 0,$$

where $a(\theta) = \begin{cases} a^+, & 0 < \theta < \beta_+, \\ a^-, & -\beta_- < \theta < 0. \end{cases}$

Singularities for intersection of discontinuity curve with boundary

Due to BC $\Phi^+(\beta_+) = \Phi^-(-\beta_-) = 0$ we obtain

$$\Phi^+ = C_+ \sin \mu(\beta_+ - \theta), \quad \Phi^- = C_- \sin \mu(\beta_- + \theta).$$

Compatibility conditions yield that

$$\begin{aligned} C_+ \sin \mu\beta_+ &= C_- \sin \mu\beta_- \\ -a^+ C_+ \mu \cos \mu\beta_+ &= a^- C_- \mu \cos \mu\beta_- . \end{aligned}$$

This system has a non-trivial solution wrt C_+, C_-, if the determinant

$$D(\mu) \equiv a^- \mu \sin \mu\beta_+ \cos \mu\beta_- + a^+ \mu \sin \mu\beta_- \cos \mu\beta_+ = 0.$$

One can show that the equation has at most two solutions on $(0, 1)$.
Finally,

$$\Psi_j(r, \theta) = \zeta(r) r^{\mu_j} \begin{cases} \sin \mu_j(\beta_+ - \theta), & 0 < \theta < \beta_+, \\ \sin \mu_j(\beta_- + \theta), & -\beta_- < \theta < 0, \end{cases}$$

where $0 < \mu_j < 1$ are roots of $D(\mu)$.

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Accuracy of variational schemes for piecewise smooth boundary

Consider piecewise linear approximations on nonregular mesh.
Let a singular point on $\partial\Omega$ be the origin, and corresp. singular function $\Psi = \zeta(r/\varepsilon) r^{\lambda} \sin \lambda\theta$, $0 < \lambda < 1$. Then solution in this neighborhood has a form $u = \gamma\Psi + w$, $w \in H^2(\Omega)$.

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Consider a triangle $\triangle$ from the mesh lying at a distance $O(h)$ from the origin. For approximate solution $\tilde{v}$

$$\|u - \tilde{v}\|_{1,\Omega} \geq \|\gamma\Psi + w - \tilde{v}\|_{1,\triangle} \geq \|\gamma\Psi + \tilde{w} - \tilde{v}\|_{1,\triangle} - \|w - \tilde{w}\|_{1,\triangle}$$
$$\geq \min_{\tilde{\phi} \in V_h} |\gamma| \|\Psi - \tilde{\phi}\|_{1,\triangle} - \|w - \tilde{w}\|_{1,\triangle}.$$

Since $\nabla\tilde{\phi}$ is constant on $\triangle$,

$$\|\gamma\Psi + w - \tilde{v}\|_{1,\triangle} \geq |\gamma| \min_{\vec{a} \in \mathbb{R}^2} \left\| \nabla\Psi - \vec{a} \right\|_{0,\triangle} - \|w - \tilde{w}\|_{1,\triangle}.$$

Accuracy of variational schemes for piecewise smooth boundary

Assume that origin is one of triangle vertices. Then (see Appendix)

$$\min_{\vec{a}\in\mathbb{R}^2} \left\| \nabla\Psi - \vec{a} \right\|_{0,\triangle} \geq Ch^{\lambda}.$$

Since $\|w - \tilde{w}\|_{1,\triangle} \leq Ch$, for $\gamma \neq 0$ and sufficiently small h an upper bound

$$\|u - \tilde{v}\|_{1,\Omega} \geq Ch^{\lambda}$$

holds.

Variational schemes for problems with discontinuous coefficients

Consider elliptic problems with smooth boundary S and pure 1st or 3rd type of BC. Here the curve of discontinuity Γ is closed and smooth, $\Gamma \cap S = \varnothing$.

Variational schemes for problems with discontinuous coefficients

Consider elliptic problems with smooth boundary S and pure 1st or 3rd type of BC. Here the curve of discontinuity Γ is closed and smooth, $\Gamma \cap S = \varnothing$.
We build the mesh Ω^h_{ex} for piecewise linear approximations using non-regular triangulations, s.t.

$$\exists \Gamma^h \subset \Omega_2 \quad \text{constructed of the sides of the triangles,}$$

$$dist(\Gamma^h, \Gamma) = O(h^2).$$

Note that

$$\|u - \tilde{v}\|_{1,\Omega} \leq C\|u - \tilde{u}\|_{1,\Omega^h_{ex}},$$

where $\tilde{u}$ is a nodal interpolant of $u \in B^2(\Omega)$.
Denote by Ω^h_1 the area bounded by the curve Γ^h. Approximation property:

$$\|u - \tilde{u}\|_{1,\Omega^h_{ex} \setminus \Omega^h_1} \leq Ch\|u\|_{2,\Omega_2}.$$

Variational schemes for problems with discontinuous coefficients

Denote by u_1 the continuation of u from Ω_1 to Ω_2, s.t. $u_1 \in H^2(\Omega)$. Note that

$$\|u - \tilde{u}\|_{1,\Omega_1^h} \leq \|u - u_1\|_{1,\Omega_1^h} + \|u_1 - \tilde{u}_1\|_{1,\Omega_1^h} + \|\tilde{u}_1 - \tilde{u}\|_{1,\Omega_1^h}.$$

Evidently, $\|u - u_1\|_{1,\Omega_1^h} = \|u - u_1\|_{1,\Omega_1^h \setminus \Omega_1}$.
$\Omega_1^h \setminus \Omega_1$ is a strip of width $O(h^2)$. Then due to a corresp. theorem

$$\begin{aligned}
\|u - u_1\|_{1,\Omega_1^h \setminus \Omega_1} &\leq Ch\|u - u_1\|_{2,\Omega_2} \\
&\leq Ch(\|u\|_{2,\Omega_2} + \|u_1\|_{2,\Omega_2}) \leq Ch(\|u\|_{2,\Omega_2} + \|u\|_{2,\Omega_1}).
\end{aligned}$$

Estimating other two terms, we obtain

$$\begin{aligned}
\|u_1 - \tilde{u}_1\|_{1,\Omega_1^h} &\leq Ch\|u\|_{2,\Omega_1}, \\
\|\tilde{u}_1 - \tilde{u}\|_{1,\Omega_1^h} &\leq Ch(\|u\|_{2,\Omega_1} + \|u\|_{2,\Omega_2}).
\end{aligned}$$

Variational schemes for problems with discontinuous coefficients

Summarizing, one derives

$$\|u - \tilde{u}\|_{1,\Omega^h_{ex}} \leq Ch(\|u\|_{2,\Omega_1} + \|u\|_{2,\Omega_2}),$$

hence

$$\|u - \tilde{v}\|_{1,\Omega} \leq Ch(\|u\|_{2,\Omega_1} + \|u\|_{2,\Omega_2}).$$

Variational schemes with additive selection of singular functions

Assume that domain Ω has two corner points on S with angles $\beta_j > \pi$. Then solution can be written as

$$u = \gamma_1 \Psi_1 + \gamma_2 \Psi_2 + w.$$

Approximate solution for the regular mesh Ω_{ex}^h we will seek in the form

$$v = \kappa_1 \Psi_1 + \kappa_2 \Psi_2 + \tilde{p},$$

where $\tilde{p}$ is a piecewise linear function from V_h.

Variational schemes with additive selection of singular functions

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$$v = \kappa_1 \Psi_1 + \kappa_2 \Psi_2 + \tilde{p},$$

where $\tilde{p}$ is a piecewise linear function from V_h.
We seek v as a solution of the integral identity

$$L_{\Omega,S}(v, \phi) = (f, \phi)_\Omega$$

for any $\phi = \mu_1 \Psi_1 + \mu_2 \Psi_2 + \tilde{\theta}$, $\tilde{\theta} \in V_h$.
In this case Galerkin system contains basis functions of V_h and Ψ_1, Ψ_2.
The following estimate holds:

$$\|u - v\|_{1,\Omega} \leq C \min_{\phi} \|u - \phi\|_{1,\Omega},$$

Variational schemes with additive selection of singular functions

Taking $\phi = \gamma_1 \Psi_1 + \gamma_2 \Psi_2 + \tilde{w}$, where $\tilde{w}$ is a nodal interpolant of w, we obtain

$$\|u - v\|_{1,\Omega} \leq C\|w - \tilde{w}\|_{1,\Omega}.$$

Results of approximation theorem in 2D:

$$\|w - \tilde{w}\|_{1,\Omega} \leq Ch\|w\|_{2,\Omega},$$
$$\|w - \tilde{w}\|_{0,\Omega} \leq Ch^2\|w\|_{2,\Omega}$$

together with the inequality $\sum_j |\gamma_j| + \|w\|_{2,\Omega} \leq C\|f\|_{0,\Omega}$ yield

$$\|u - v\|_{1,\Omega} \leq \tilde{C}h\|f\|_{0,\Omega},$$
$$\|u - v\|_{0,\Omega} \leq \tilde{C}h^2\|f\|_{0,\Omega}.$$

Solution of Galerkin system with singular basis functions

Matrix L of the system

$$\begin{aligned} L_{\Omega,S}(v, \Psi_1) &= (f, \Psi_1)_\Omega, \\ L_{\Omega,S}(v, \Psi_2) &= (f, \Psi_2)_\Omega, \\ L_{\Omega,S}(v, \phi_{k_i}) &= (f, \phi_{k_i})_\Omega. \end{aligned}$$

is dense at 1st and 2nd rows.

Solution of Galerkin system with singular basis functions

Matrix L of the system

$$L_{\Omega,S}(v,\Psi_1) = (f,\Psi_1)_\Omega,$$
$$L_{\Omega,S}(v,\Psi_2) = (f,\Psi_2)_\Omega,$$
$$L_{\Omega,S}(v,\phi_{k_i}) = (f,\phi_{k_i})_\Omega.$$

is dense at 1st and 2nd rows.

We apply orthogonal factorization:

$$v = \tilde{w}_0 + k_1 w_1 + k_2 w_2,$$

where $L_{\Omega,S}(w_1,\tilde{w}_0) = L_{\Omega,S}(w_2,\tilde{w}_0) = L_{\Omega,S}(w_2,w_1) = 0.$

Solution of Galerkin system with singular basis functions

Let $w_1 \equiv \Psi_1 - \tilde{q}_1$, where $\tilde{q}_1 \in V_h$ solves the equation

$$L_{\Omega,S}(\Psi_1 - \tilde{q}_1, \tilde{\theta}) = 0 \tag{10}$$

for any $\tilde{\theta} \in V_h$.

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for any $\tilde{\theta} \in V_h$.

Then define $w_2 \equiv \Psi_2 + \mu w_1 - \tilde{q}_2$, s.t.

$$L_{\Omega,S}(\tilde{q}_2, \tilde{\theta}) = L_{\Omega,S}(\Psi_2, \tilde{\theta}) \tag{11}$$

for any $\tilde{\theta} \in V_h$, and

$$\mu = -\frac{L_{\Omega,S}(\Psi_2 - \tilde{q}_2, w_1)}{L_{\Omega,S}(w_1, w_1)}$$

to satisfy $L_{\Omega,S}(w_2, w_1) = 0$.

Solution of Galerkin system with singular basis functions

3rd equation is to find $\tilde{w}_0 \in V_h$:

$$L_{\Omega,S}(\tilde{w}_0, \tilde{\theta}) = (f, \tilde{\theta})_\Omega \tag{12}$$

for any $\tilde{\theta} \in V_h$.
Coefficients k_1, k_2 can be found from the expressions

$$k_1 = \frac{(f, w_1)_\Omega - L_{\Omega,S}(\tilde{w}_0, w_1)}{L_{\Omega,S}(w_1, w_1)},$$

$$k_2 = \frac{(f, w_2)_\Omega - L_{\Omega,S}(\tilde{w}_0, w_2) - k_1 L_{\Omega,S}(w_1, w_2)}{L_{\Omega,S}(w_2, w_2)}.$$

Solution of Galerkin system with singular basis functions

Assembling matrices for (10), (11), (12) requires the computation of $L_{\Omega,S}(\Psi,\Psi)$, $L_{\Omega,S}(\Psi,\phi_{k_i})$, $(f,\Psi)_\Omega$.
On the mesh triangles in the neighborhood of the corner points one has to evaluate the integrals of the form

$$\int_\triangle \alpha\left(\frac{\partial\Psi}{\partial x_1}\right)^{i_1}\left(\frac{\partial\Psi}{\partial x_2}\right)^{i_2}\,d\Omega,$$

where α is a linear function, $1\leq i_1+i_2\leq 2$.
Since in the polar coordinates $\Psi=\zeta(r)r^\lambda\cos\lambda\theta$, this double integrals reduce to multiple integrals of the terms as

$$r^\nu\cos^{m_1}\lambda\theta\sin^{m_2}\lambda\theta\cos^{n_1}\theta\sin^{n_2}\theta.$$

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Appendix

Now we prove that

$$\min_{\vec{a}\in\mathbb{R}^2} \left\| \nabla\Psi - \vec{a} \right\|_{0,\triangle} \geq Ch^{\lambda},$$

where $\Psi = \zeta(r/\varepsilon) r^{\lambda} \sin \lambda\theta$, $0 < \lambda < 1$.
Note that for sufficiently small $h > 0$ we have $\Psi = r^{\lambda} \sin \lambda\theta$.
Differentiating the function $\left\| \nabla\Psi - \vec{a} \right\|_{0,\triangle}^2$ w.r.t. $\vec{a}$, one obtains the minimum for

$$\vec{a}_* = \frac{1}{\int_\triangle 1\ dx} \Big(\int_\triangle \frac{\partial\Psi}{\partial x_1}\ dx, \int_\triangle \frac{\partial\Psi}{\partial x_2}\ dx \Big).$$

Further we use the relations

$$\frac{\partial\Psi}{\partial x_1} = \cos\theta \frac{\partial\Psi}{\partial r} - \frac{1}{r} \sin\theta \frac{\partial\Psi}{\partial\theta},$$
$$\frac{\partial\Psi}{\partial x_2} = \sin\theta \frac{\partial\Psi}{\partial r} + \frac{1}{r} \cos\theta \frac{\partial\Psi}{\partial\theta}.$$

Appendix

For convenience we will also consider circle sectors $\sigma \subset \triangle \subset \bar{\sigma}$. Then

$$\int_{\bar{\sigma}} F(x)\ dx \geq \int_{\triangle} F(x)\ dx \geq \int_{\sigma} F(x)\ dx$$

for any $F(x) \geq 0$ and, switching to polar coordinates, we have

$$\int_{\sigma} F(x)\,dx = \int_0^{ch} dr \int_{\theta_1}^{\theta_2} d\theta\, rF(r,\theta),\ \int_{\bar{\sigma}} F(x)\,dx = \int_0^{\bar{c}h} dr \int_{\theta_1}^{\theta_2} d\theta\, rF(r,\theta).$$

for some $0 < c < \bar{c}$.

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for some $0 < c < \bar{c}$.

Expressions for $\frac{\partial \Psi}{\partial x_1}$, $\frac{\partial \Psi}{\partial x_2}$ allow us to estimate

$$\left| \int_{\triangle} \frac{\partial \Psi}{\partial x_1}\ dx \right| \leq \int_{\bar{\sigma}} \left| \frac{\partial \Psi}{\partial x_1} \right|\ dx \leq C^0_{\bar{\sigma}} \int_0^{ch} r^{\lambda}\ dr \leq C_{\bar{\sigma}} h^{\lambda+1},$$

as well as for $\frac{\partial \Psi}{\partial x_2}$.

Appendix

Taking into account that $\int_{\triangle} 1 \, dx = O(h^2)$, last inequality means: $|\vec{a}_*| = O(h^{\lambda-1})$.

Expressions for $\frac{\partial \Psi}{\partial x_1}$, $\frac{\partial \Psi}{\partial x_2}$, which can be expressed as $G_i(\theta) r^{\lambda-1}$, $i = 1, 2$ with certain trigonometric functions $G_i(\theta)$, also yield that on a fixed circle sector $\sigma_1 \subset \sigma$ (with polar angles $\theta \in (\theta_1, \theta_2)$ away from the roots of $G_i(\theta)$ and radius $\underline{c}h$ for some $0 < \underline{c} < c$) the estimates

$$\left|\frac{\partial \Psi}{\partial x_1}\right| \geq c_0 r^{\lambda-1}, \ \left|\frac{\partial \Psi}{\partial x_2}\right| \geq c_0 r^{\lambda-1}$$

hold.

Then due to the estimate for $|\vec{a}_*|$ one can choose such a circle sector $\sigma_2 \subset \sigma_1$ (with radius αh for some $\alpha > 0$) that for a given $c_1 < c_0$:

$$\left|\nabla \Psi - \vec{a}_*\right|_2 \geq c_1 r^{\lambda-1}.$$

Appendix

Finally, we obtain

$$\left\|\nabla\Psi - \vec{a}_*\right\|_{0,\triangle}^2 \geq \left\|\nabla\Psi - \vec{a}_*\right\|_{0,\sigma_2}^2$$
$$= \int_0^{\alpha h} dr \int_{\theta_1}^{\theta_2} d\theta \; r\left|\nabla\Psi - \vec{a}_*\right|_2^2$$
$$\geq \int_0^{\alpha h} dr \int_{\theta_1}^{\theta_2} d\theta \; r c_1^2 r^{2\lambda-2} \geq c_3 \int_0^{\alpha h} r^{2\lambda-1} \, dr = C^2 h^{2\lambda},$$

which proves the required lower estimate for $\left\|\nabla\Psi - \vec{a}_*\right\|_{0,\triangle}$.