Green's formula and density results

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Seminar on Numerical Analysis

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Motivation

- extend trace theorem for polygonal domain Ω, when $u \notin H^2(\Omega)$
- corresponding generalization of Green's formula
- prove density results for spaces with boundary conditions (for polygonal domains)

Standard Green formulas

Theorem

Let Ω be a bounded Lipschitz open subset of $\mathbb{R}^n$. Then for any $u, v \in H^1(\Omega)$

$$\int_\Omega v D_i u \; dx + \int_\Omega u D_i v \; dx = \int_\Gamma \gamma u \; \gamma v \; \nu^i \; d\sigma,$$

where $D_i = \frac{\partial}{\partial x_i}$ and ν^i is the i-th component of exterior unit normal vector ν.

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where $D_i = \frac{\partial}{\partial x_i}$ and ν^i is the i-th component of exterior unit normal vector ν.

- For $u \in H^1(\Omega)$, $v \in H^2(\Omega)$ we have 'half Green formula'

$$\int_\Omega u \triangle v \; dx + \int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Gamma \gamma u \; \gamma\Big(\frac{\partial v}{\partial \nu}\Big) \; d\sigma$$

- for $u, v \in H^2(\Omega)$ we have the full Green formula

$$\int_\Omega u \triangle v \; dx - \int_\Omega v \triangle u \; dx = \int_\Gamma \gamma u \; \gamma\Big(\frac{\partial v}{\partial \nu}\Big) \; d\sigma - \int_\Gamma \gamma v \; \gamma\Big(\frac{\partial u}{\partial \nu}\Big) \; d\sigma.$$

Trace theorem: Formulation

Define

$$D(\triangle, L^2(\Omega)) = \{v \in L^2(\Omega) \colon \triangle v \in L^2(\Omega)\},$$

where $\triangle v$ is understood in the distributional sense.
This is a Hilbert space for the norm

$$v \mapsto \left(\|v\|^2 + \|\triangle v\|^2\right)^{1/2}.$$

Similarly to [Lions-Magenes, 1968] one can prove that $H^2(\Omega)$ is dense in $D(\triangle, L^2(\Omega))$.

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Similarly to [Lions-Magenes, 1968] one can prove that $H^2(\Omega)$ is dense in $D(\triangle, L^2(\Omega))$.

Theorem

Let Ω be a bounded polygonal open subset of $\mathbb{R}^2$. Then the mapping $v \mapsto \left\{\gamma_j v,\ \gamma_j \frac{\partial v}{\partial \nu_j}\right\}$, which is defined for $v \in H^2(\Omega)$, has a unique continuous extension as an operator

$$D(\triangle, L^2(\Omega)) \to \widetilde{H}^{-1/2}(\Gamma_j) \times \widetilde{H}^{-3/2}(\Gamma_j).$$

Trace theorem: Proof

The full Greeen formula implies for any $u, v \in H^2(\Omega)$

$$\Big| \sum_j \Big\{ \int_{\Gamma_j} \gamma_j u \; \gamma_j\Big(\frac{\partial v}{\partial \nu_j}\Big) \; d\sigma - \int_{\Gamma_j} \gamma_j v \; \gamma_j\Big(\frac{\partial u}{\partial \nu_j}\Big) \; d\sigma \Big\} \Big| \leq K \|u\|_{2,\Omega},$$

where K depends on v and can be chosen as
$K = \big(\|v\|^2_{0,\Omega} + \|\triangle v\|^2_{0,\Omega}\big)^{1/2}$.

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where K depends on v and can be chosen as
$K = \big(\|v\|^2_{0,\Omega} + \|\triangle v\|^2_{0,\Omega}\big)^{1/2}$.
For fixed j, consider

$$U = \{u \in H^2(\Omega) \colon \gamma_k u = \gamma_k\big(\frac{\partial u}{\partial \nu_k}\big) = 0 \text{ on } \Gamma_k \text{ for all } k \neq j\}.$$

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The full Greeen formula implies for any $u, v \in H^2(\Omega)$

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$$U = \{u \in H^2(\Omega)\colon \gamma_k u = \gamma_k\big(\frac{\partial u}{\partial \nu_k}\big) = 0 \text{ on } \Gamma_k \text{ for all } k \ne j\}.$$

Then for any $u \in U$

$$\Big|\int_{\Gamma_j} \gamma_j u \; \gamma_j\Big(\frac{\partial v}{\partial \nu_j}\Big)\; d\sigma - \int_{\Gamma_j} \gamma_j v \; \gamma_j\Big(\frac{\partial u}{\partial \nu_j}\Big)\; d\sigma\Big| \le K\|u\|_{2,\Omega}.$$

Trace theorem: Proof

It is shown (in the previous report) that $u \mapsto \left\{ f_{j,0} = \gamma_j u,\ f_{j,1} = \gamma_j\left(\frac{\partial u}{\partial \nu_j}\right) \right\}$ maps U onto $\widetilde{H}^{3/2}(\Gamma_j) \times \widetilde{H}^{1/2}(\Gamma_j)$ defined by

a) $f_{j,0}(S_j) = f_{j,0}(S_{j-1}) = 0$;

b) $\frac{\partial f_{j,0}}{\partial \tau_j} \equiv 0$ and $f_{j,1} \equiv 0$ at S_j and S_{j-1}.

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Remark

Recall that functions φ_j and φ_{j+1} defined on Γ_j and Γ_{j+1} respectively, are equivalent at S_j: $\varphi_j \equiv \varphi_{j+1}$, if

$$\int_0^{\delta_j} |\varphi_j(x_j(-\sigma)) - \varphi_{j+1}(x_j(\sigma))|^2 \, d\sigma/\sigma < +\infty.$$

When φ_j and φ_{j+1} are Hölder continuous near S_j, it follows that $\varphi_j(S_j) = \varphi_{j+1}(S_j)$.

Trace theorem: Proof

Since the linear mapping $u \mapsto \left\{ \varphi = \gamma_j u,\ \psi = \gamma_j\left(\frac{\partial u}{\partial \nu_j}\right) \right\}$ is 'onto' and for any $(\varphi, \psi) \in \widetilde{H}^{3/2}(\Gamma_j) \times \widetilde{H}^{1/2}(\Gamma_j)$ their prototype u uniformly satisfies

$$\|u\|_U = \|u\|_{2,\Omega} \leq C \|(\varphi, \psi)\|_{\widetilde{H}^{3/2}(\Gamma_j) \times \widetilde{H}^{1/2}(\Gamma_j)},$$

the following linear form

$$L(\varphi, \psi) = \int_{\Gamma_j} \varphi \gamma_j \left(\frac{\partial v}{\partial \nu_j} \right) \, d\sigma - \int_{\Gamma_j} \psi \gamma_j v \, d\sigma$$

is continuous on $\widetilde{H}^{3/2}(\Gamma_j) \times \widetilde{H}^{1/2}(\Gamma_j)$.

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is continuous on $\widetilde{H}^{3/2}(\Gamma_j) \times \widetilde{H}^{1/2}(\Gamma_j)$.

Due to the density of $H^2(\Omega)$ in $D(\triangle, L^2(\Omega))$ and uniform boundedness of $L(\varphi, \psi)$ w.r.t. $K = \|v\|_{D(\triangle, L^2(\Omega))}$, there exists a (unique continuous) extension of $v \mapsto \left\{ \gamma_j v,\ \gamma_j \frac{\partial v}{\partial \nu_j} \right\}$ as an operator

$$D(\triangle, L^2(\Omega)) \to \widetilde{H}^{-1/2}(\Gamma_j) \times \widetilde{H}^{-3/2}(\Gamma_j).$$

Generalizing Green formula

It is shown in [Lions-Magenes, 1968] that the full Green formula still holds for $v \in D(\triangle, L^2(\Omega))$ and $u \in H^2(\Omega)$ when

- Ω is bounded and has C^∞ boundary;
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- Ω is bounded and has C^∞ boundary;
- $\int_\Gamma$ are understood as duality brackets.

Problem of extending this result for polygonal Ω:

- $\int_\Gamma$ as duality brackets are meaningful only if

$$\gamma_j u \in \widetilde{H}^{3/2}(\Gamma_j), \quad \gamma_j\left(\frac{\partial u}{\partial \nu_j}\right) \in \widetilde{H}^{1/2}(\Gamma_j)$$

  and this is not true for all $u \in H^2(\Omega)$.

Theorem for generalized Green formula

Theorem

Let Ω be a bounded polygonal open subset of $\mathbb{R}^2$. Then for any $v \in D(\triangle, L^2(\Omega))$ and $u \in H^2(\Omega)$ such that

$$\gamma_j u \in \widetilde{H}^{3/2}(\Gamma_j), \quad \gamma_j\Big(\frac{\partial u}{\partial \nu_j}\Big) \in \widetilde{H}^{1/2}(\Gamma_j) \text{ for all } j,$$

we have

$$\int_\Omega u\triangle v\ dx - \int_\Omega v\triangle u\ dx = \sum_j \Big(\langle \gamma_j u, \gamma_j(\frac{\partial v}{\partial \nu_j})\rangle - \langle \gamma_j v, \gamma_j(\frac{\partial u}{\partial \nu_j})\rangle\Big).$$

Theorem for generalized Green formula: Proof

The equality holds for any $u, v \in H^2(\Omega)$.

For any fixed u (that satisfies the conditions of the theorem) from the proof of trace theorem one can see that both sides of the equality are continuous in the norm of $D(\triangle, L^2(\Omega))$ w.r.t. v.

Then the statement follows from the density of $H^2(\Omega)$ in $D(\triangle, L^2(\Omega))$.

Other types of trace theorems

Assumption $v \in D(\triangle, L^2(\Omega))$ is made to ensure the continuity of

$$v \mapsto \int_\Omega u \triangle v \; dx - \int_\Omega v \triangle u \; dx$$

for $u \in H^2(\Omega)$. Since we have $u \in C(\overline{\Omega})$ $(n = 2)$, we can just assume $v \in D(\triangle, L^1(\Omega)) = \{w \in L^2(\Omega) \colon \triangle w \in L^1(\Omega)\}$.

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Theorem (Grisvard (1985))

Let Ω be a bounded polygonal open subset of $\mathbb{R}^2$. Then $H^2(\Omega)$ is dense in $E(\triangle, L^p(\Omega)) = \{v \in H^1(\Omega) \colon \triangle v \in L^p(\Omega)\}$ $(p > 1)$ and mapping $v \mapsto \gamma_j\left(\frac{\partial v}{\partial \nu_j}\right)$ has unique continuous extensions as an operator from $E(\triangle, L^p(\Omega))$ into $\widetilde{H}^{-1/2}(\Gamma_j)$.
In addition, one has

$$\int_\Omega u \triangle v \; dx = -\int_\Omega \nabla u \cdot \nabla v \; dx + \sum_j \langle \gamma_j u, \gamma_j \left(\frac{\partial v}{\partial \nu_j}\right) \rangle$$

for any $v \in E(\triangle, L^p(\Omega))$, $u \in H^1(\Omega)$ such that $\gamma_j u \in \widetilde{H}^{1/2}(\Gamma_j)$ for all j.

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Motivation: density with boundary conditions

While analyzing homogeneous BVP

- one has to approximate Sobolev spaces with homogeneous boundary conditions by smoother functions with the same boundary conditions;
- in the case of smooth domains the corresponding density results are the consequences of trace results.

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- in the case of smooth domains the corresponding density results are the consequences of trace results.

In the case of polygons

- similar results are in general difficult to prove;
- we focus on the few cases of direct use.

First density result

For a mixed BVP for the Laplace equation on a bounded polygonal set $\Omega \subset \mathbb{R}^2$ we use

$$V = \{u \in H^1(\Omega) \colon \gamma_j u = 0 \text{ on } \Gamma_j,\ j \in \mathcal{D}\}.$$

Theorem

The space $H^m(\Omega) \cap V$ is dense in V for any $m > 1$.

Density result: Proof

We prove an equivalent statement:

- ⋆ any continuous linear form on V that vanishes on $H^m(\Omega) \cap V$, actually vanishes everywhere.

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One can represent a linear form on V as

$$l(v) = \langle S, v - \rho\gamma v \rangle + \langle g, \gamma v \rangle,$$

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$$l(v) = \langle S, v - \rho\gamma v \rangle + \langle g, \gamma v \rangle,$$

where $S \in H^{-1}(\Omega)$, $g \in R^*$, ρ is a right inverse of γ.

- l vanishes on $H^m(\Omega) \cap V \;\Rightarrow\; l$ vanishes on $\mathcal{D}(\Omega)$
- $\Rightarrow\; \langle S, v \rangle = 0$ on $\mathcal{D}(\Omega) \;\Rightarrow\; S = 0.$

Density result: Proof

l depends only on $\gamma v \;\Rightarrow$

- we need only to check that $T^m(\Gamma)$ is dense in $T^1(\Gamma)$,
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As it was proven (in the previous report),

- $T^1(\Gamma)$ is a subspace of $\Big\{ \prod_{j \in \mathcal{N}} H^{1/2}(\Gamma_j) \Big\}$,
- $\Rightarrow$ any element of $T^1(\Gamma)$ can be denoted by $\{g_j\}_{j \in \mathcal{N}}$.

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Since $g_j = 0$ for $j \in \mathcal{D}$, it is known that

$$g_{j+1} \equiv g_j \text{ at } S_j \text{ for every } j. \tag{1}$$

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As it was proven,

- $T^m(\Gamma)$ is a subspace of $\{\prod_{j \in \mathcal{N}} H^{m-1/2}(\Gamma_j)\}$ defined by

$$g_{j+1}(S_j) = g_j(S_j) \text{ for every } j. \tag{2}$$

Density result: Proof

We characterize the condition (1) by

- the function $\sigma \mapsto g_{j+1}(x_j(\sigma)) - g_j(x_j(-\sigma))$ belongs to $\widetilde{H}^{1/2}(\mathbb{R}_+)$ near zero when $j \in \mathcal{N}^2$;
- the function $\sigma \mapsto g_{j+1}(x_j(\sigma))$ belongs to $\widetilde{H}^{1/2}(\mathbb{R}_+)$ near zero when $j \in \mathcal{D}$ and $j+1 \in \mathcal{N}$;
- the function $\sigma \mapsto g_j(x_j(-\sigma))$ belongs to $\widetilde{H}^{1/2}(\mathbb{R}_+)$ near zero when $j \in \mathcal{N}$ and $j+1 \in \mathcal{D}$.

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- the function $\sigma \mapsto g_{j+1}(x_j(\sigma))$ belongs to $\widetilde{H}^{1/2}(\mathbb{R}_+)$ near zero when $j \in \mathcal{D}$ and $j+1 \in \mathcal{N}$;
- the function $\sigma \mapsto g_j(x_j(-\sigma))$ belongs to $\widetilde{H}^{1/2}(\mathbb{R}_+)$ near zero when $j \in \mathcal{N}$ and $j+1 \in \mathcal{D}$.

Then the density of $T^m(\Gamma)$ in $T^1(\Gamma)$ follows as

- ▷ the condition (2) is clearly fulfilled when $g_j \in \mathcal{D}(\mathbb{R}_+)$;
- ▷ $\mathcal{D}(\mathbb{R}_+)$ is dense in $\widetilde{H}^{1/2}(\mathbb{R}_+)$.

Second density result

In studying a mixed BVP for the Laplace equation on a bounded polygon $\Omega \subset \mathbb{R}^2$ will be used a space of strong solutions

$$V^2(\Omega) = \Big\{ u \in H^2(\Omega) \colon \gamma_j u = 0 \text{ on } \Gamma_j,\ j \in \mathcal{D} \text{ and } \gamma_j\big(\frac{\partial u}{\partial \nu_j}\big) = 0 \text{ on } \Gamma_j,\ j \in \mathcal{N} \Big\}.$$

Theorem

The space $H^m(\Omega) \cap V^2(\Omega)$ is dense in $V^2(\Omega)$ for any $m > 1$.

Density for strong solutions: Proof

Equivalent statement:

- ⋆ any continuous linear form on $V^2(\Omega)$ that vanishes on $V^m(\Omega) = H^m(\Omega) \cap V^2(\Omega)$, actually vanishes everywhere.

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- ⋆ $H^m(\Omega)$ is a direct sum of $H_0^m(\Omega)$ and the image $Z^m(\Omega)$ of the operator $\gamma = \{\gamma_j(\frac{\partial^l}{\partial\nu_j^l})\}_{1\leq j\leq N, 0\leq l\leq m-1}$.

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One can represent a linear form on V as

$$l(v) = \langle S, v - \rho\gamma v \rangle + \langle g, \gamma v \rangle,$$

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As it was proven, $Z^2(\Gamma)$ is a subspace of $\prod_j H^{3/2}(\Gamma_j) \times H^{1/2}(\Gamma_j)$, whose elements $\{g_j, h_j\}_{j\in\mathcal{N}}$ are defined by

$$
\begin{aligned}
&g_j = 0 \text{ on } \Gamma_j \text{ for } j \in \mathcal{D}\\
&h_j = 0 \text{ on } \Gamma_j \text{ for } j \in \mathcal{N}\\
&g_j(S_j) = g_{j+1}(S_j)\\
&g_j' \equiv -g_{j+1}' \cos\omega_j + h_{j+1} \sin\omega_j \text{ at } S_j \text{ for every } j\\
&h_j \equiv -h_{j+1} \cos\omega_j + g_{j+1}' \sin\omega_j \text{ at } S_j \text{ for every } j
\end{aligned}
$$

Density for strong solutions: Proof

Lemma

The image of $H^m(\Omega)$ by the mapping

$$u \mapsto \{\gamma_j u = g_j,\ \gamma_j \frac{\partial u}{\partial \nu_j} = h_j\}_{1 \le j \le N}$$

is the subspace of $\prod_j H^{m-1/2}(\Gamma_j) \times H^{m-3/2}(\Gamma_j)$ defined by

$$g_j(S_j) = g_{j+1}(S_j) \tag{3}$$

$$g_j'(S_j) = -g_{j+1}'(S_j)\cos\omega_j + h_{j+1}(S_j)\sin\omega_j \tag{4}$$

$$h_j(S_j) = -h_{j+1}(S_j)\cos\omega_j + g_{j+1}'(S_j)\sin\omega_j \tag{5}$$

$$-g_j''(S_j)\cos\omega_j - h_j'(S_j)\sin\omega_j = -g_{j+1}''(S_j)\cos\omega_j + h_{j+1}'(S_j)\sin\omega_j \tag{6}$$

when $m \ge 4$ and

$$-g_j''\cos\omega_j - h_j'\sin\omega_j \equiv -g_{j+1}''\cos\omega_j + h_{j+1}'\sin\omega_j \text{ at } S_j$$

when $m = 3$.

Density for strong solutions: Proof

Then we describe $Z^m(\Gamma)$ as a subspace of $\prod_j H^{m-1/2}(\Gamma_j) \times H^{m-3/2}(\Gamma_j)$ defined by

$$g_j = 0 \text{ on } \Gamma_j \text{ for } j \in \mathcal{D}$$
$$h_j = 0 \text{ on } \Gamma_j \text{ for } j \in \mathcal{N}$$

and (3), (4), (5), (6),

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Proving density of $Z^m(\Gamma)$ in $Z^2(\Gamma)$ is carried out by

- considering the conditions for g_j, h_j near each corner S_j;
- applying density of $\mathcal{D}(\mathbb{R}_+)$ in $\widetilde{H}^{1/2}(\mathbb{R}_+)$.