01 Show that every linear second order partial differential equation

$$-\big(a(x)\,u'(x)\big)' + b(x)\,u'(x) + c(x)\,u(x) = f(x),$$

with $a \in \mathcal{C}^1(0,1)$ and $b,\, c \in \mathcal{C}(0,1)$ can be rewritten in the form

$$\bar{a}(x)\,u''(x) + \bar{b}(x)\,u'(x) + c(x)\,u(x) = f(x),$$

and find suitable functions $\bar{a} \in \mathcal{C}^1(0,1)$ and $\bar{b} \in \mathcal{C}(0,1)$. Show also the reverse direction.

02 Derive the variational formulations of the two following boundary value problems:

(a)
$$\begin{cases} -u''(x) + u(x) &= f(x) \qquad \text{for } x \in (0,1) \\ u(0) &= g_0 \\ u(1) &= g_1 \end{cases}$$

(b)
$$\begin{cases} -u''(x) + u(x) &= f(x) \qquad \text{for } x \in (0,1) \\ u(0) &= g_0 \\ u'(1) &= g_1 - \alpha_1\,u(1) \end{cases}$$

In particular, specify the spaces V_g, and V_0, the bilinear form $a(\cdot,\cdot)$, and the linear form $\langle F, \cdot\rangle$.
Hint for (b): Perform integration by parts as usual, substitute $u'(1)$ due to the Robin boundary condition, and collect the bilinear and linear terms accordingly.

03 Consider the boundary value problem

$$\begin{aligned} -\big(a(x)\,u'(x)\big)' &= 1 \qquad \text{for } x \in (0,1), \\ u(0) &= 0, \\ a(1)\,u'(1) &= 0, \end{aligned} \tag{1.1}$$

where $a(x) = \sqrt{2x - x^2}$. Justify that $u(x) = \sqrt{2x - x^2}$ is a *classical* solution of (1.1), i. e., $u \in X := \mathcal{C}^2(0,1) \cap \mathcal{C}^1(0,1] \cap \mathcal{C}[0,1]$. Furthermore, show that

$$\int_0^1 |u'(x)|^2\,dx = \infty\,.$$

Note: This example shows that $u \notin H^1(0,1)$, i. e., u is no *weak* solution.

04 Consider the piecewise constant coefficient function $a \in L^\infty(0,1)$,

$$a(x) = \begin{cases} a_1 & \text{for } x \in \left[0, \frac{1}{2}\right], \\ a_2 & \text{for } x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

with positive constants $a_1 \neq a_2$. Derive a variational formulation for the boundary value problem

$$\begin{aligned} -a(x)\, u''(x) &= f(x) \qquad \text{for } x \in (0,1) \setminus \{\tfrac{1}{2}\}, \\ u(0) &= g_1, \qquad u(1) = g_2, \end{aligned}$$

together with the *transmission conditions*

$$u(\tfrac{1}{2}^-) = u(\tfrac{1}{2}^+), \qquad a_1\, u'(\tfrac{1}{2}^-) = a_2\, u'(\tfrac{1}{2}^+),$$

where $w(\frac{1}{2}^-)$ and $w(\frac{1}{2}^+)$ denote the left sided and right sided limit of a function w, respectively.
Hint: Integration by parts is only valid on subintervals!

05 Let the sequence $(u_k)_{k\in\mathbb{N}}$ of functions be defined by

$$u_k(x) = \begin{cases} 2x & \text{for } x \in \left[0, \frac{1}{2} - \frac{1}{2k}\right], \\ 1 - \frac{1}{2k} - 2k\left(x - \frac{1}{2}\right)^2 & \text{for } x \in \left(\frac{1}{2} - \frac{1}{2k}, \frac{1}{2} + \frac{1}{2k}\right), \\ 2(1-x) & \text{for } x \in \left[\frac{1}{2} + \frac{1}{2k}, 1\right]. \end{cases}$$

Show that $u_k \in \mathcal{C}^1[0,1]$. Let u be defined by

$$u(x) = \begin{cases} 2x & \text{for } x \in \left[0, \frac{1}{2}\right], \\ 2(1-x) & \text{for } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Find out if u, $u_k \in H^1(0,1)$ or not and justify your answer. Calculate $\|u_k - u\|_{H^1(0,1)}$ (maybe with a little help from Mathematica/Maple) or find a suitable bound for it in order to show that

$$\lim_{k\to\infty} \|u_k - u\|_{H^1(0,1)} = 0.$$

Use these results to show that $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}^1[0,1]$ with respect to the H^1-norm, but that there exists no limit in $\mathcal{C}^1[0,1]$.

06 Show that there exists **no** function $w \in L^2(0,1)$ such that

$$\varphi(\tfrac{1}{2}) = \int_0^1 w(x)\varphi(x)\mathrm{d}x \qquad \text{for all } \varphi \in \mathcal{C}_0^\infty(0,1).$$

Hint: Consider the sequence of test functions

$$\varphi_n(x) := \begin{cases} e^{1 - \frac{1}{1-n^2(1-2x)^2}} & \text{for } |1-2x| < \frac{1}{n}, \\ 0 & \textit{else} \end{cases} \in \mathcal{C}_0^\infty(0,1) \qquad \text{for } n \in \mathbb{N}, n \geq 2.$$