Element stiffness matrices:

$$(K_h \underline{v}_h, \underline{w}_h)_{\ell^2} = a(v_h, w_h) = \sum_{k=1}^{n_h} \sum_{i,j=1}^{n_h} \underbrace{\int_{T_k} \varphi_i'(x)\, \varphi_j'(x)\, dx}_{\substack{=0 \text{ if } i \notin \{k-1,k\} \\ \text{or } j \notin \{k-1,k\}}} v_j\, w_i$$

$$= K_h^{(1)}\, w_1\, v_1 + \sum_{k=2}^{n_h} \left(K_h^{(k)} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

where

$$K_h^{(1)} = \int_{T_1} \varphi_1'(x)^2\, dx =: K_{11}^{(1)}$$

$$K_h^{(k)} \stackrel{k \geq 1}{=} \begin{bmatrix} \int_{T_k} \varphi_{k-1}'(x)^2\, dx & \int_{T_k} \varphi_{k-1}'(x)\, \varphi_k'(x)\, dx \\ \int_{T_k} \varphi_k'(x)\, \varphi_{k-1}'(x)\, dx & \int_{T_k} \varphi_k'(x)^2\, dx \end{bmatrix} =: \begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix}$$

$$(K_h \underline{v}_h, \underline{w}_h)_{\ell^2} = K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

To obtain entries K_{ij} of K_h, we use that for

$$v_j = 1, \text{ other entries of } \underline{v}_h \text{ zero}$$
$$w_i = 1, \text{ other entries of } \underline{w}_h \text{ zero}$$

we have

$$(K_h \underline{v}_h, \underline{w}_h)_{\ell^2} \;=\; K_{ij}$$

General formula:

$$(K_h \underline{v}_h, \underline{w}_h)_{\ell^2} = K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

Case $i = 1$, $j = 1$: $v_1 = w_1 = 1$

$$K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

$k = 2$

Hence:

$$K_{11} = K_{11}^{(1)} + K_{00}^{(2)}$$

General formula:

$$(K_h \underline{v}_h, \underline{w}_h)_{\ell^2} = K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

Case $i = 1, j = 2$:

$$v_2 = w_1 = 1$$

$$K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

$$k = 2$$

Hence:

$$K_{12} = K_{01}^{(2)}$$

General formula:

$$(K_h \underline{v}_h, \underline{w}_h)_{\ell^2} = K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

Case $i = 2$, $j = 1$:

$$v_1 = w_2 = 1$$

$$K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

$$k = 2$$

Hence:

$$K_{21} = K_{10}^{(2)}$$

General formula:

$$(K_h\,\underline{v}_h,\,\underline{w}_h)_{\ell^2} = K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

Case $i = 2$, $j = 2$:

$$v_2 = w_2 = 1$$

$$K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

$k = 2$

$k = 3$

Hence:

$$K_{22} = K_{11}^{(2)} + K_{00}^{(3)}$$

and so on ...

General formula:

$$(K_h \underline{v}_h, \underline{w}_h)_{\ell^2} = K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

Case $i = n_h - 1$, $j = n_h - 1$:

$$v_{n_h-1} = w_{n_h-1} = 1$$

$$K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

$$k = n_h - 1$$

$$k = n_h$$

Hence:

$$K_{n_h-1,n_h-1} = K_{11}^{(n_h-1)} + K_{00}^{(n_h)}$$

General formula:

$$(K_h\,\underline{v}_h,\,\underline{w}_h)_{\ell^2} = K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

Case $i = n_h - 1,\ j = n_h$:

$$v_{n_h} = w_{n_h-1} = 1$$

$$K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

$$k = n_h$$

Hence:

$$K_{n_h-1,n_h} = K_{01}^{(n_h)}$$

General formula:

$$(K_h\,\underline{v}_h,\,\underline{w}_h)_{\ell^2} = K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

Case $i = n_h$, $j = n_h$:

$$v_{n_h} = w_{n_h} = 1$$

$$K_{11}^{(1)} w_1 v_1 + \sum_{k=2}^{n_h} \left(\begin{bmatrix} K_{00}^{(k)} & K_{01}^{(k)} \\ K_{10}^{(k)} & K_{11}^{(k)} \end{bmatrix} \begin{bmatrix} v_{k-1} \\ v_k \end{bmatrix}, \begin{bmatrix} w_{k-1} \\ w_k \end{bmatrix} \right)_{\ell^2}$$

$$k = n_h$$

Hence:

$$K_{n_h,n_h} = K_{11}^{(n_h)}$$

Element stiffness matrices (1D model problem)

Global stiffness matrix:

$$
\begin{bmatrix}
K_{11}^{(1)} + K_{00}^{(2)} & K_{01}^{(2)} & 0 & \cdots & \cdots & \cdots & 0 \\
K_{10}^{(2)} & K_{11}^{(2)} + K_{00}^{(3)} & K_{01}^{(3)} & \ddots & & & \vdots \\
0 & K_{10}^{(3)} & K_{11}^{(3)} + K_{00}^{(4)} & K_{01}^{(4)} & \ddots & & \vdots \\
\vdots & \ddots & K_{10}^{(4)} & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & K_{01}^{(n_h-1)} & 0 \\
\vdots & & & \ddots & K_{10}^{(n_h-1)} & K_{11}^{(n_h-1)} + K_{00}^{(n_h)} & K_{01}^{(n_h)} \\
0 & \vdots & \vdots & \vdots & 0 & K_{10}^{(n_h)} & K_{11}^{(n_h)}
\end{bmatrix}
$$

Assembling

Start: empty matrix (zero)

$$\begin{bmatrix} 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & \\ & 0 & 0 & 0 & & & \\ & & 0 & \ddots & \ddots & & \\ & & & \ddots & \ddots & 0 & \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \end{bmatrix}$$

Assembling

Add contribution from element 1

$$
\begin{bmatrix}
\blacksquare & 0 & & & & & \\
0 & 0 & 0 & & & & \\
 & 0 & 0 & 0 & & & \\
 & & 0 & \ddots & \ddots & & \\
 & & & \ddots & \ddots & 0 & \\
 & & & & 0 & 0 & 0 \\
 & & & & & 0 & 0
\end{bmatrix}
$$

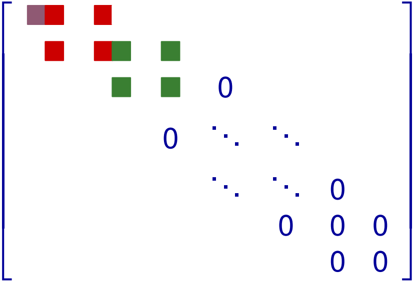
Assembling

Add contribution from element 2

$$\begin{bmatrix} \blacksquare & \blacksquare & & & & & \\ \blacksquare & \blacksquare & 0 & & & & \\ & 0 & 0 & 0 & & & \\ & & 0 & \ddots & \ddots & & \\ & & & \ddots & \ddots & 0 & \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \end{bmatrix}$$

Assembling

Add contribution from element 3



and so on ...