Convergence Analysis of the Improved Euler Method

$$u_0 \in \mathbb{R}^n \text{ given}$$
$$u_{j+1} = u_j + \tau_j \underbrace{f\big(t_j + \tfrac{\tau_j}{2}, u_j + \tfrac{\tau_j}{2} f(t_j, u_j)\big)}_{=\phi(t_j, u_j, \tau_j)}$$

Stability analysis

We show that (13) (f Lipschitz, constant L) implies (18) (ϕ Lipschitz, constant Λ):

$$\begin{aligned}
\|\phi(t,v,\tau) - \phi(t,w,\tau)\| &= \|f(t+\tfrac{\tau}{2}, v + \frac{\tau}{2} f(t,v)) - f(t+\tfrac{\tau}{2}, w + \frac{\tau}{2} f(t,w)\| \\
&\overset{(13)}{\le} L\,\|v + \tfrac{\tau}{2} f(t,v) - v - \tfrac{\tau}{2} f(t,w)\| \\
&\le L\big[\|v-w\| + \frac{\tau}{2} \underbrace{\|f(t,v) - f(t,w)\|}_{\overset{(13)}{\le} L\|v-w\|} \\
&\le L(1+\tfrac{\tau}{2}L)\,\|v-w\| \le \underbrace{L(1+\tfrac{\tau_{\max}}{2}L)}_{=:\Lambda}\,\|v-w\|
\end{aligned}$$

Now Theorem 2.29 implies: improved Euler is stable.

Consistency analysis

(a) Classical consistency analysis of the local error by Taylor expansion
For simplicity assume that $n = 1$ and that f (and therefore also u) is sufficiently smooth

$$\begin{aligned}
d_\tau(t+\tau) &= u(t+\tau) - \big[u(t) + \tau\,\phi(t, u(t), \tau)\big] \\
&= u(t+\tau) - u(t) - \tau\, f\big(t + \tfrac{\tau}{2}, u(t) + \tfrac{\tau}{2} f(t, u(t))\big) \\
&= u(t) + \tau\, u'(t) + \tfrac{\tau^2}{2} u''(t) + \tfrac{\tau^3}{6} u'''(t) + \mathcal{O}(\tau^4) - u(t) \\
&\quad - \tau\Big[\big(f + \tfrac{\tau}{2} f_t + \tfrac{\tau}{2} f_u\, f + \tfrac{\tau^2}{4} f_{tt} + \tfrac{\tau^2}{2} f_{tu}\, f + \tfrac{\tau^2}{4} f_{uu}\, f^2\big)_{(t,\,u(t))} + \mathcal{O}(\tau^3)\Big] \\
&= \tau\Big[\underbrace{u'(t) - f(t, u(t))}_{=0}\Big] + \tfrac{\tau^2}{2}\Big[\underbrace{u''(t) - \big(f_t - f_u\, f\big)_{(t,\,u(t))}}_{=(*)=0}\Big] + \\
&\quad + \tau^3\Big[\tfrac{1}{6} u'''(t) - \tfrac{1}{4}\big(f_{tt} + 2 f_{tu}\, f + f_{uu}\, f^2\big)_{(t,\,u(t))}\Big] + \mathcal{O}(\tau^4)
\end{aligned}$$

where

$$0 = \frac{d}{dt}\Big[u'(t) - f(t, u(t))\Big] = u''(t) - \big(f_t(t, u(t)) + f_u(t, u(t))\, \underbrace{u'(t)}_{=f(t,u(t))}\,\big) = (*)$$

(b) Estimate of the consistency error
One can show that under appropriate smoothness assumptions on f,

$$\exists K > 0 : \ \|\psi_\tau(t)\|_{Y_\tau} \le K\,\tau^2$$

i.e., the improved Euler method has **consistency order** 2 (in the sense of Def. 2.25).

Convergence now follows from Lemma 2.27 (stability+consistency $\implies$ convergence):

$$\|e_\tau\|_{X_\tau} \le e^{L(1+\frac{\tau_{\max}}{2}L)T}\, K\,\tau^2$$