

# Consistency Analysis of Implicit Runge-Kutta methods

In principle, the consistency analysis fits into the framework of one-steps methods, using Taylor expansion. However, it is harder because  $\phi(t, u, \tau)$  is defined *implicitly*.

## 1. Implicit Euler method

$$u_{j+1} = u_j + \tau_j \phi(t_j, u_j, \tau_j)$$

with  $\phi(t, u, \tau) := f(t + \tau, \gamma(t, u, \tau))$  where

$$\gamma(t, u, \tau) = u + \tau f(t + \tau, \gamma(t, u, \tau)) \quad (1)$$

Taylor series expansion of the local error at  $\tau = 0$ :

$$\begin{aligned} d_\tau(t + \tau) &= u(t + \tau) - [u(t) + \tau \phi(t, u(t), \tau)] \\ &= u(t) + \tau u'(t) + \frac{\tau^2}{2} u''(t) + \mathcal{O}(\tau^3) - \\ &\quad - [u(t) + \tau (\phi(t, u(t), 0) + \tau \phi_\tau(t, u(t), 0) + \mathcal{O}(\tau^2))] \end{aligned}$$

We need the partial derivative

$$\phi_\tau(t, u(t), 0) = f_t(t, \gamma(t, u(t), 0)) \cdot 1 + f_u(t, \gamma(t, u(t), 0)) \gamma_\tau(t, u(t), 0)$$

Value given via implicit equation:

$$\gamma(t, u, 0) = u$$

For the derivative, first differentiate (1) w.r.t.  $\tau$ :

$$\gamma_\tau(t, u, \tau) = f(t + \tau, \gamma(t, u, \tau)) + \tau [f_t(t + \tau, \gamma(t, u, \tau)) \cdot 1 + f_u(t, \gamma(t, u, \tau)) \gamma_\tau(t, u, \tau)]$$

for  $\tau = 0$  this implies

$$\gamma_\tau(t, u, 0) = f(t, \gamma(t, u, 0)) = f(t, u)$$

Hence,

$$\begin{aligned} \phi(t, u(t), 0) &= f(t, \gamma(t, u(t), 0)) = f(t, u(t)) = u'(t) \\ \phi_\tau(t, u(t), 0) &= f_t(t, \gamma(t, u(t), 0)) + f_u(t, \gamma(t, u(t), 0)) \gamma_\tau(t, u(t), 0) \\ &= f_t(t, u(t)) + f_u(t, u(t)) f(t, u(t)) = u''(t) \end{aligned}$$

and so

$$\begin{aligned} d_\tau(t + \tau) &= u(t) + \tau u'(t) + \frac{\tau^2}{2} u''(t) + \mathcal{O}(\tau^3) - \\ &\quad - [u(t) + \tau (u'(t) + \tau u''(t) + \mathcal{O}(\tau^2))] = -\frac{\tau^2}{2} u''(t) + \mathcal{O}(\tau^3) \end{aligned}$$

Leading error term of the consistency error  $\psi_\tau(u)(t)$  is  $-(\tau/2) u''(t)$ .

*Consistency order 1.*

## 2. $\theta$ -method

$$\psi_\tau(u)(t) = \left(\frac{1}{2} - \theta\right)\tau u''(t) + \mathcal{O}(\tau^2)$$

For  $\theta \neq 1/2$ : *Consistency order 1*

For  $\theta = 1/2$  (implicit trapezoidal rule): *Consistency order 2*

## 3. Implicit midpoint rule

$$\psi_\tau(u)(z) = \mathcal{O}(\tau^2)$$

*Consistency order 2* (1-stage method!)

### In general:

An  $s$ -stage **implicit** Runge-Kutta method has maximal consistency order  $2s$ .

Such methods are called Runge-Kutta methods of *Gauss type*, they are based on Gaussian quadrature rules.

*Example:* The implicit midpoint rule is of Gauss type.