The fixed point iteration behind the Lax-Milgram proof

Recall that for suitable τ and for arbitrary initial values $u_0 \in V$, the fixed point iteration

$$(13) \qquad u_{k+1} \;=\; \psi_\tau(u_k) \;=\; u_k - \tau\,\mathcal{R}(G - B\,u_k)$$

converges to the solution $u \in W$ of

$$(11) \qquad b(u,\, v) \;=\; \langle G,\, v\rangle \quad \forall v \in W.$$

We decode one iteration step of (13):

1. Form the *residual* $R_k := G - B\,u_k \quad \in W^*$
2. Form the *correction* $w_k := \mathcal{R}\,R_k \quad \in W$
3. Form the next approximant $u_{k+1} := u_k + \tau\,w_k$

In Step 2, we need Riesz' isomorphism $\mathcal{R} : W^* \to W$. Recall, however, that Riesz' proof is *non-constructive.*

We rewrite Step 2 using the definitions of $\mathcal{R}$ and $\mathcal{I}$:

$$\begin{aligned}
& & w_k &= \mathcal{R}(G - B\,u_k) \\
&\overset{\mathcal{R}^{-1}=\mathcal{I}}{\Longleftrightarrow} & \mathcal{I}\,w_k &= G - B\,u_k \\
&\Longleftrightarrow & \langle \mathcal{I}\,w_k,\, v\rangle &= \langle G,\, v\rangle - \langle B\,u_k,\, v\rangle \quad \forall v \in W \\
&\overset{\text{Def. }\mathcal{I}}{\Longleftrightarrow} & (w_k,\, v)_W &= \langle G,\, v\rangle - b(u_k,\, v) \qquad \forall v \in W
\end{aligned}$$

Hence, Step 2 is equivalent to the variational problem:

$$(14) \qquad \text{Find } w_k \in W\text{:} \quad (w_k,\, v)_W \;=\; \underbrace{\langle G,\, v\rangle - b(u_k,\, v)}_{=\langle R_k,\, v\rangle} \qquad \forall v \in W.$$

"Unfortunately", for our concrete setting with the PDE background ($b \mapsto a$, $G \mapsto \widehat{F}$, $W \mapsto V_0$), Problem (14) corresponds to a boundary value problem of a PDE (*exercise/Tutorial:* find out which one). Therefore, carrying out step 2 is (almost) as difficult as solving the original problem (6).

Conclusion: So far, the fixed point iteration (13) is only of theoretical interest; it cannot be turned into a numerical scheme *directly*. However, we will be able to reuse the ideas from above in later sections.