Definition 1.48
Let $(\cdot,\cdot)$ be an inner product in $\mathbb{R}^n$ with associated norm $\|\cdot\|$ and let $A \in \mathbb{R}^{n\times n}$.

1. A is **self-adjoint** w.r.t. $(\cdot,\cdot)$ iff
$$(A\,y,\,z) \;=\; (y,\,A\,z) \qquad \forall y, z \in \mathbb{R}^n\,.$$
Note: A is self-adjoint w.r.t. $(\cdot,\cdot)_{\ell^2}$ (Euclidean inner product) $\iff$ A is symmetric ($A = A^\top$).

2. The **spectrum** of A (the finite set of eigenvalues) is defined by
$$\sigma(A) := \big\{\lambda \in \mathbb{C} : \exists x \in \mathbb{C}^n \setminus \{0\} : A\,x = \lambda\,x\big\}.$$
If A is self-adjoint w.r.t. $(\cdot,\cdot)$, then $\sigma(A) \subset \mathbb{R}$. We define
$$\lambda_{\min}(A) := \min_{\lambda\in\sigma(A)} \lambda, \qquad \lambda_{\max}(A) := \max_{\lambda\in\sigma(A)} \lambda.$$

3. Let A be self-adjoint w.r.t. $(\cdot,\cdot)$.
   (a) A is **positive semi-definite** ($A \geq 0$) iff $\quad (A\,y,\,y) \geq 0 \quad \forall y \in \mathbb{R}^n$
   (b) A is **positive definite** $\quad (A > 0)$ iff $\quad (A\,y,\,y) > 0 \quad \forall y \in \mathbb{R}^n \setminus \{0\}$

Lemma 1.49

(i) $A \geq 0 \iff \forall\lambda \in \sigma(A) : \lambda \geq 0 \iff \lambda_{\min}(A) \geq 0$

(ii) $A > 0 \iff \forall\lambda \in \sigma(A) : \lambda > 0 \iff \lambda_{\min}(A) > 0$

(iii) $\lambda_{\min}(A) = \inf\limits_{y\in\mathbb{R}^n\setminus\{0\}} \underbrace{\dfrac{(A\,y,\,y)}{(y,\,y)}}_{\text{Rayleigh quotient}} \qquad \lambda_{\max}(A) = \sup\limits_{y\in\mathbb{R}^n\setminus\{0\}} \dfrac{(A\,y,\,y)}{(y,\,y)}$

Proof: (i), (ii) easy, (iii): exercise

Lemma 1.50 If A is self-adjoint and positive definite then
$$\|A\| \;=\; \sup_{y\in\mathbb{R}^n\setminus\{0\}} \frac{|(A\,y,\,y)|}{(y,\,y)} \;=\; \lambda_{\max}(A), \qquad \|A^{-1}\| \;=\; \frac{1}{\lambda_{\min}(A)}\,.$$

Hence, the **condition number** $\kappa(A) := \|A\|\|A^{-1}\| = \dfrac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$.

Proof: Since A is self-adjoint,
$$\|y\| \;=\; \sup_{y\in\mathbb{R}^n\setminus\{0\}} \frac{|(A\,y,\,y)|}{(y,\,y)}\,.$$
Since A is positive definite, $|(A\,y,\,y)| = (A\,y,\,y)$. With Lemma 1.49 (iii), this implies $\|A\| = \lambda_{\max}(A)$. The properties of A imply now that A^{-1} indeed exists and is self adjoint w.r.t. $(\cdot,\cdot)$. From the fact $\lambda \in \sigma(A) \iff \lambda^{-1} \in \sigma(A^{-1})$, it follows that $\|A^{-1}\| = \lambda_{\max}(A^{-1}) = 1/\lambda_{\min}(A)$. □

Lemma 1.51 Let A and C be self-adjoint w.r.t. $(\cdot,\cdot)$ and let $C > 0$.
Then $C^{-1}A$ is self-adjoint w.r.t. the inner product
$$(y,\,z)_C := (C\,y,\,z)\,.$$