Overview: classical vs. variational formulation

Classical formulation

Find $u \in X := C^2(0,1) \cap C^1(0,1] \cap C[0,1]$:

(1) $\quad (a\,u')' + b\,u' + c\,u \;=\; f \qquad \text{in } (0,1)$

(2) $\quad u(0) \;=\; g_0$

(3) $\quad a(1)\,u'(1) \;=\; g_1$

where

$$(7) \quad \begin{cases} a \in C^1(0,1) \cap C(0,1] \\ b,\, c \in C(0,1) \end{cases}$$

Variational formulation

Find $u \in V_g := \{v \in H^1(0,1) : v(0) = g_0\}$:

(6) $\quad a(u,\, v) \;=\; \langle F,\, v\rangle$

for all $v \in V_0 := \{v \in H^1(0,1) : v(0) = 0\}$

where

$$a(u,\, v) := \int_0^1 a\,u'\,v' + b\,u'\,v + c\,u\,v\,dx$$

$$\langle F,\, v\rangle := \int_0^1 f\,v\,dx + g_1\,v(1)$$

and $a,\, b,\, c \in L^\infty(0,\, 1)$

WARNINGS:

in general
classical solution $\not\Longrightarrow$ weak solution

$$\left.\begin{array}{l} u \text{ classical solution} \\ u \in C^2[0,1] \\ a,\, b,\, c \in L^\infty(0,1) \end{array}\right\} \Longrightarrow u \text{ weak solution}$$

in general
classical solution $\not\Longleftarrow$ weak solution

$$u \text{ classical solution} \quad \Longleftarrow \left\{\begin{array}{l} u \text{ weak solution} \\ u \in X \\ (7) \end{array}\right.$$