Comparison:

FEM system $K_h \, \underline{u}_h = \underline{f}_h$ for quasi-uniform mesh, d-dimensional model problem

operations	$d=1$	$d=2$	$d=3$
Gauss, optimal reordering, band structure	$\mathcal{O}(n_h)$	$\mathcal{O}(n_h^{3/2})$	$\mathcal{O}(n_h^2)$
method of steepest descent	$\mathcal{O}(n_h^3)$	$\mathcal{O}(n_h^2)$	$\mathcal{O}(n_h^{5/3})$
CG	$\mathcal{O}(n_h^2)$	$\mathcal{O}(n_h^{3/2})$	$\mathcal{O}(n_h^{4/3})$

Implementation of CG

Lemma 1.63 implies

(i) $\alpha_k = \dfrac{(r_k, p_k)}{(Ap_k, p_k)} = \dfrac{(r_k, r_k)}{(Ap_k, p_k)}$

(ii) $\alpha_{k-1}(r_k, Ap_{k-1}) = (r_k, \underbrace{\alpha_{k-1}Ap_{k-1}}_{-r_k+r_{k-1}}) = -(r_k, r_k)$

(iii) $\beta_{k-1} = -\dfrac{(r_k, Ap_{k-1})}{(Ap_{k-1}, p_{k-1})} \overset{(ii)}{=} \dfrac{(r_k, r_k)}{\alpha_{k-1}(Ap_{k-1}, p_{k-1})} \overset{(i)}{=} \dfrac{(r_k, r_k)}{(r_{k-1}, r_{k-1})}$

Algorithm 1.66 (CG)

$\boldsymbol{x_0}$ given

$\boldsymbol{r_0 = b - A\,x_0}$

$\boldsymbol{k = 0}$

repeat

$$\boldsymbol{p_k} = \begin{cases} \boldsymbol{r_0} & \text{if } \boldsymbol{k=0} \\ \boldsymbol{r_k + \beta_{k-1} p_k} \quad \text{with } \boldsymbol{\beta_{k-1}} = \dfrac{\boldsymbol{(r_k, r_k)}}{\boldsymbol{(r_{k-1}, r_{k-1})}} & \text{else} \end{cases}$$

$$\boldsymbol{x_{k+1} = x_k + \alpha_k\, p_k} \quad \text{with } \boldsymbol{\alpha_k} = \dfrac{\boldsymbol{(r_k, p_k)}}{\boldsymbol{(A\,p_k, p_k)}}$$

$$\boldsymbol{r_{k+1} = r_k - \alpha_k\, A\, p_k} \qquad (= b - A\,x_{k+1})$$

$$\boldsymbol{k = k+1}$$

until *stopping criterion fulfilled, e.g.* $\boldsymbol{\|r_k\| \le \varepsilon\, \|r_0\|}$
