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70 Show the following estimate from the lecture:

$$\frac{1}{\tau_j}\int_{t_j}^{t_{j+1}} (u'(s), v)\, ds \;-\; (\,(1-\theta)\, u'(t_j) + \theta\, u'(t_{j+1}),\, v\,)$$
$$\leq \int_{t_j}^{t_{j+1}} |(u''(\sigma), v)|\, d\sigma \qquad \forall u \in C^2\ \forall \theta \in [0,1]$$

71 Derive the stability function $R(z)$ of the θ-method,

$$R(z) = 1 + \frac{z}{1-\theta z}\ .$$

Then determine the stability region

$$S = \{z \in \mathbb{C}\ :\ |R(z)| \leq 1\}$$

for the cases $0 \leq \theta < 1/2$, $\theta = 1/2$ and $1/2 < \theta \leq 1$ separately and give center and radius of the corresponding circle (see also your lecture notes).

Hint: You may use e.g. Mathematica to ease these computations.

72 Consider the 1D model problem with a Dirichlet boundary condition in $x = 0$. The entries of the mass matrix,

$$[M_h]_{i,j} = \int_0^1 \varphi_j(x)\varphi_i(x)dx\ ,$$

can be approximated by the trapezoidal rule, giving an approximation

$$[\bar{M}_h]_{i,j} := \sum_{k=1}^{n_h} \frac{h_k}{2}\left(\varphi_j(x_{k-1})\varphi_i(x_{k-1}) + \varphi_j(x_k)\varphi_i(x_k)\right)\ .$$

Show:

- $\bar{M}_h$ is diagonal
- $[\bar{M}_h]_{i,i} = \sum_{j=1}^{n_h}[M_h]_{i,j} \qquad$ for $i \geq 2$.

73 Write functions or overload operators to implement

- A copy constructor or copy operator for `SMatrix`
- The multiplication of a matrix with a scalar
- The addition of two matrices

such that you can compute a Matrix $A = M_h + \tau\theta K_h$.

$\boxed{74}$ Write a function which implements one step of the θ- method, following the interface

```
void thetaMethod(double theta, double tau,
  const Vector& f_old, const Vector& f_new,
  const SMatrix& M, const SMatrix& K,
  const Vector& u, Vector& u_new)
```

where

`theta` ... parameter θ
`tau` ... time step size τ_j
`f_old` ... load vector at time t_j
`f_new` ... load vector at time t_{j+1}
`M,K` ... mass- and stiffness matrix, respectively
`u,u_new` ... solution vectors at t_j and t_{j+1}, respectively

Hint: Use your direct solver for tridiagonal matrices to compute $\underline{\phi}_{h,j+1}$ in

$$[M_h + \tau\theta K_h]\, \underline{\phi}_{h,j+1} = (1-\theta)\underline{f}_h(t_j) + \theta\underline{f}_h(t_{j+1}) - K_h\underline{u}_{h,j} \ .$$

The new solution is given by

$$\underline{u}_{h,j+1} = \underline{u}_{h,j} + \tau_j\underline{\phi}_{h,j+1} \ .$$

$\boxed{75}$ Solve the parabolic problem in exercise $\boxed{65+66}$ with the θ- method for fixed time steps τ and a uniform FEM discretization in space with step size h (so, $n_h + 1 = 1/h + 1$ nodes!). Compare the results of explicit ($\theta = 0$) and implicit ($\theta = 1$) Euler and the implicit trapezoidal rule ($\theta = 1/2$) for different values of h and τ on the time interval $[0, T] = [0, 1]$. Compute the l_2-norm of the solution vector $\underline{u}_{h,m}$ at the last time step $t_m = T = 1$.

h	1/8	1/8	1/10	1/10
τ	1/330	1/390	1/550	1/600
implicit Euler: $\|\underline{u}_{h,m}\|_{l_2}$				
explicit Euler: $\|\underline{u}_{h,m}\|_{l_2}$				
impl. trapezoidal: $\|\underline{u}_{h,m}\|_{l_2}$				

Print some of the solutions (e.g. columns 1 and 3) to files and plot the results, eg. in Mathematica or Matlab.

When does explicit Euler give reasonable results? Do your observations comply with the corresponding theory?