Numerical Methods for Partial Differential Equations WS 2012 / 13
Tutorial 12 Monday, January 14, 2013, 10.15–11.45, S2 054

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64 (a) Write functions to assemble the mass matrix as presented in the lecture, analogously to the corresponding functions to assemble the stiffness matrix:

```
void ElementMassMatrix (double xa, double xb, Mat22& elMat);
void AssembleMassMatrix (const Mesh& mesh, SMatrix& mat);
```

(b) Write a function which implements one step of the Explicit Euler method, following the interface

```
void explicitEuler(double tau, const Vector& f, const SMatrix& M,
                   const SMatrix& K, const Vector& u, Vector& u_new)
```

where
`tau` ... time step size τ_j
`f` ... load vector at time t_j
`M,K` ... mass- and stiffness matrix, respectively
`u,u_new` ... solution vectors at t_j and t_{j+1}, respectively

Hint: The new solution is given by `u_new = u + tau * w`, where w is computed by solving the linear system `M w = f - K u`. Use the direct solver which you implemented in exercise **32**.

65+66 Solve the following parabolic problem with the Explicit Euler method with fixed time steps $\tau = 1/320$ and a FEM discretization in space with $h = 1/8$:

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) &= f(t,x) & \\ u(t,0) = u(t,1) &= 0 & \forall t \in [0,T] \\ u(0,x) &= u_0(x) & \forall x \in [0,1] \end{aligned}$$

where $f(t,x) = 2, \quad u_0(x) = 2\sin(x\pi)$ and $T = 0.25$.

In this special case you can reuse much of your code for elliptic problems:

- Create an equidistant mesh of $[0,1]$ with 8 elements for x.
- Assemble mass matrix M_h and stiffness matrix K_h
- Assemble the load vectors $\underline{g}_h$ and $\underline{f}_h$ for the given functions u_0 and f.
- Implement both Dirichlet Boundary Conditions for M_h, $\underline{g}_h$.
- Implement both Dirichlet Boundary Conditions for K_h, $\underline{f}_h$.
- Compute the semi-discrete initial value $\underline{u}_{h,0}$ at $t = 0$ by solving $M_h\,\underline{u}_{h,0} = \underline{g}_h$ using your direct solver of exercise **64**.
- For each time step $t_j = j\tau$ compute $\underline{u}_{h,j+1}$ from $\underline{u}_{h,j}$ using Explicit Euler.

Print the solution to a file and plot the result, eg. in Mathematica or Matlab. You might want to consult the plotting tips on the website.

Comments:

- Don't despair if your solution looks awful for high values of t (The Explicit Euler method is not stable!). It should however look plausible up to $t = 0.1$.
- In the next tutorial, you will be asked to do something similar for the Implicit Euler method, which will be easier if your Explicit Euler works!

In the following two exercises, let X be a Banach space with norm $\|\cdot\|$ and consider the general initial value problem to find $u : [0,\infty) \to X$ such that

$$\begin{aligned} u'(t) &= f(t, u(t)) \qquad \forall t \geq 0, \\ u(0) &= u_0 \,, \end{aligned}$$

with $u_0 \in X$ and $f : [0,\infty) \times X \to X$ given.

67 Assume that there exists a constant $L > 0$ such that

$$\|f(t,v) - f(t,w)\| \;\leq\; L\,\|v-w\| \qquad \forall t \geq 0 \qquad \forall v, w \in X. \tag{12.1}$$

Show that for each given $t_{j+1} > 0$ and $u_j \in X$, there exists a unique solution $\boldsymbol{v}$ to the implicit equation

$$\boldsymbol{v} \;=\; u_j + \tau_j\, f(t_{j+1}, \boldsymbol{v}), \tag{12.2}$$

if $0 < \tau_j < 1/L$. *Hint:* use Banach's fixed point theorem.

68 Assume that X is a Hilbert space with the inner product $(\cdot,\cdot)$ and that

$$\big(f(t,v) - f(t,w),\, v - w\big) \;\leq\; 0 \qquad \forall t \geq 0 \qquad \forall v, w \in X \tag{12.3}$$

holds additionally to (12.1). Show that for each given $\tau_j > 0$, t_{j+1} and $u_j \in X$, there exists a unique solution $\boldsymbol{v}$ to the implicit equation (12.2).
Hint: apply Banach's fixed point theorem to the equivalent equation

$$\boldsymbol{v} \;=\; G(\boldsymbol{v}) := (1-\rho)\boldsymbol{v} + \rho\big(u_j + \tau_j\, f(t_{j+1}, \boldsymbol{v})\big),$$

where you have to choose the parameter $\rho \in (0,1)$ such that G is a contraction.

69 Apply the Implicit Euler method with a given stepsize τ to the following differential equation:

$$u'(t) = -50\, u(t) \qquad u(0) = 1$$

You may use a programming language of your choice (like Mathematica or Matlab). Plot the solutions for $\tau = \frac{1}{60}, \frac{1}{30}, \frac{1}{20}, \frac{1}{10}$ on $[0,T] = [0,1]$. How do the results relate to observations in the lecture?

Remark: You will not need to reuse or expand your code in future tutorials. Just make sure you get the results and think about them!