58 Prove formula (19) from the lecture, i.e. show that

$$\|\psi_{j+1}(u)\| \leq \int_{t_j}^{t_{j+1}} \|u''(t)\| dt$$

holds for $j = 0, \dots, m-1$ and for $u \in C^2([0,T], \mathbb{R}^n)$.

59 Assume that f fulfills the Lipschitz condition (13), i.e.

$$\|f(t,v) - f(t,w)\| \;\leq\; L\,\|v-w\| \qquad \forall t \in [0,T] \quad \forall v,w \in \mathbb{R}^n\,.$$

Furthermore, let the sequences (u_j) and (v_j) be given according to the (perturbed) explicit Euler method:

$$\left.\begin{array}{rcl} u_{j+1} &=& u_j + \tau_j\, f(t_j, u_j) \\ v_{j+1} &=& v_j + \tau_j\,[f(t_j, v_j) + y_{j+1}] \end{array}\right\} \qquad \forall j = 0, \dots, m-1,$$

and $v_0 = u_0 + y_0$ for given (but arbitrary) values u_0 and $y_0, \dots, y_m \in \mathbb{R}^n$. Show that then,

$$\|u_j - v_j\| \;\leq\; e^{(t_j - t_0)L}\,\|y_0\| + \frac{1}{L}\left(e^{(t_j-t_0)L} - 1\right) \max_{k=1,\dots,j} \|y_k\|$$

for all $\tau > 0$.
Hint: Show and use $e^{(t_j - t_k)L}\,\tau_{k-1} \leq \int_{t_{k-1}}^{t_k} e^{(t_j - s)L}\, ds$.

60 Recall that the definitions of consistency, stability, and convergence depend on the norms $\|\cdot\|_{X_\tau}$ and $\|\cdot\|_{Y_\tau}$. In this exercise, we replace $\|\cdot\|_{Y_\tau}$ by $\|\cdot\|_{X_\tau}$.

Use Exercise **59** to show an estimate of the form

$$\|e_\tau\|_{X_\tau} \;\leq\; C\,\|\psi_\tau(u)\|_{X_\tau}$$

for the explicit Euler method with a stability constant C independent of τ. Furthermore, show that for exact solutions $u \in C^2([0,T], \mathbb{R}^n)$,

$$\|\psi_\tau(u)\|_{X_\tau} \;\leq\; K\,\tau$$

with $K = \max_{s \in [0,T]} \|u''(s)\|$.

61 Consider the general explicit 2-stage Runge-Kutta method

$$\begin{array}{rcl} g_1 &=& u_j \\ g_2 &=& u_j + \tau_j\, a_{2,1}\, f(t_j,\, g_1) \\ u_{j+1} &=& u_j + \tau_j\left[b_1\, f(t_j,\, g_1) + b_2\, f(t_j + c_2\,\tau_j,\, g_2)\right] \end{array}$$

with coefficients $a_{2,1}$, b_1, b_2 and c_2 for the approximate solution of the initial value problem to find $u : [0,T] \to \mathbb{R}$ such that

$$\begin{array}{rcl} u'(t) &=& f(t,\, u(t)) \qquad \forall t \in (0,T), \\ u(t) &=& u_0\,, \end{array}$$

where $u_0 \in \mathbb{R}$ is given and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is sufficiently smooth.

A Taylor series expansion of the *local error* of the form

$$d_\tau(t+\tau) \;=\; A_0 + A_1\,\tau + A_2\,\tau^2 + A_3\,\tau^3 + \mathcal{O}(\tau^4),$$

with the expressions $A_0, \ldots, A_3$ depending only on $a_{2,1}$, b_1, b_2, c_2, f, and its derivatives, but not on τ, is provided in the KUSSS-system.

Find necessary conditions on the coefficients $a_{2,1}$, b_1, b_2, and c_2 such that the consistency order of the method is at least 2, i.e. such that for all sufficiently smooth functions f,

$$A_0 \;=\; A_1 \;=\; A_2 \;=\; 0.$$

Is it possible to get consistency order 3?

62 Consider the classical Runge-Kutta method of order 4,

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

Show that this method has the stability function

$$R(z) \;=\; 1 + z + \frac{1}{2}\,z^2 + \frac{1}{6}\,z^3 + \frac{1}{24}\,z^4\,.$$

63 Apply the Explicit Euler method with a given stepsize τ to the following differential equation:

$$u'(t) = -50\;u(t) \qquad u(0) = 1$$

You may use a programming language of your choice (like Mathematica or Matlab). Plot the solutions for $\tau = \frac{1}{60}, \frac{1}{30}, \frac{1}{26}, \frac{1}{24}$ on $[0,T] = [0,1]$. How do the results relate to observations in the lecture?

Remark: You will not need to reuse or expand your code in future tutorials. Just make sure you get the results and think about them!