Let V, H be separable Hilbert spaces. We consider the abstract problem: find $u \in H^1\big((0,T), V; H\big)$ such that

$$\left.\begin{array}{rcll} \underbrace{\frac{d}{dt}(u(t),\, v)_H}_{=\langle u'(t), v\rangle} + a(u(t),\, v) &=& \langle F(t),\, v\rangle & \forall v \in V \quad \forall t \in (0,T) \text{ a.e.}, \\ u(0) &=& u_0 & \text{in } H, \end{array}\right\} \tag{10.1}$$

for given $u_0 \in H$ and $F \in L^2\big((0,T), V^*\big)$, where (V, H, V^*) is an evolution triple, i. e.,

- $V \subset H$,
- there exists a constant $c > 0$ such that $\|v\|_H \le c\,\|v\|_V$ for all $v \in V$,
- V is dense in H.

Recall that the weak time derivative $u' \in L^2((0,T),\, V^*)$ fulfills

$$\int_0^T \varphi(t)\,\langle u'(t),\, v\rangle\, dt \;=\; -\int_0^T \varphi'(t)\,(u(t),\, v)_H\, dt \qquad \forall v \in V\ \forall \varphi \in C_0^\infty(0,\, T).$$

52 Show that for all $\lambda \in \mathbb{R}$: the function $u \in H^1\big((0,T), V; H\big)$ is a solution of (10.1) *if and only if* $u_\lambda \in H^1\big((0,T), V; H\big)$ solves

$$\left.\begin{array}{rcll} \frac{d}{dt}(u_\lambda(t),\, v)_H + a_\lambda(u_\lambda(t),\, v) &=& \langle F_\lambda(t),\, v\rangle & \forall v \in V \quad \forall t \in (0,T) \text{ a.e.}, \\ u_\lambda(0) &=& u_0 & \text{in } H, \end{array}\right\} \tag{10.2}$$

where

$$u_\lambda(t) = e^{-\lambda t} u(t), \qquad a_\lambda(w,v) = a(w,v) + \lambda(w,v)_H\,, \qquad F_\lambda(t) = e^{-\lambda t}F(t).$$

Hint: use the definition of the weak time derivative $u'(t)$ to compute $u'_\lambda(t)$.

53 Let $a : V \times V \to \mathbb{R}$ be a bounded bilinear form. Show that if constants $\lambda \in \mathbb{R}$ and $\mu_1 > 0$ exist such that the so-called *Gårding inequality*

$$a(v,v) + \lambda\,\|v\|_H^2 \;\ge\; \mu_1\,\|v\|_V^2 \qquad \forall v \in V$$

holds, then the problem (10.1) is uniquely solvable for any $F \in L^2((0,T); V^*)$ and $u_0 \in H$.

Hint: Use Exercise **52** and Theorem 2.8.

Now, consider the bilinear form

$$a(w,\, v) := \int_0^1 a(x)\,\frac{\partial w}{\partial x}(x)\,\frac{\partial v}{\partial x}(x) + b(x)\,\frac{\partial w}{\partial x}(x)\,v(x) + c(x)\,w(x)\,v(x)\,dx$$

in $H^1(0,1)$ with a, b, $c \in L^\infty(0,1)$, where $a_0 := \inf_{x\in(0,1)} a(x) > 0$. Show the Gårding inequality for this bilinear form.

Hint: Choose λ such that the assumptions of Excercise **11** hold for the bilinear form $a_\lambda(w,\, v) := a(w,\, v) + \lambda\,(w, v)_{L^2(0,1)}$.

54 Let $C^1([0,T],V)$ denote the space of continuous functions in $[0,T]$ with values in the Hilbert space V that have a continuous *classical* derivative, i. e., for $v \in C^1([0,T],V)$ the limit

$$v'(t) := \lim_{\tau \to 0} \frac{1}{\tau}\big(v(t+\tau) - v(t)\big)$$

exists for all $t \in [0,T]$ and the function $v' : [0,T] \to V$ is continous.

Show that for all $s,\, t \in [0,T]$ and for all $v \in C^1([0,T],V)$:

$$\frac{1}{2}\big(v(t),\, v(t)\big)_H \;=\; \frac{1}{2}\big(v(s),\, v(s)\big)_H + \int_s^t \big(v'(\sigma),\, v(\sigma)\big)_H \, d\sigma. \qquad (10.3)$$

Hint: Prove and use the identity

$$\frac{1}{2}\Big[\big(v(\sigma),\, v(\sigma)\big)_H\Big]' \;=\; \big(v'(\sigma),\, v(\sigma)\big)_H \,.$$

55 Prove the statement of Lemma 2.7 for continuously differentiable functions, i. e., show that there exists a constant $C > 0$ such that

$$\max_{t\in[0,\,T]} \|v(t)\|_H \;\le\; C\,\|v\|_{H^1((0,T),V;H)} \qquad \forall v \in C^1([0,T],V).$$

Hint: Integrate identity (10.3) w.r.t. s over $[0,T]$. Note that $\|v'\|^2_{L^2((0,T),V^*)}$ is the integral over the (square of the) V^*-norm of the mapping $w \mapsto (v'(t), w)_H$. Show and use that

$$(v'(\sigma),\, w)_H \;\le\; \|v'(\sigma)\|_{V^*}\,\|w\|_V \;\le\; \frac{1}{2}\Big[\|v'(\sigma)\|^2_{V^*} + \|w\|^2_{V^*}\Big] \qquad \forall w \in V.$$

56 Consider the Courant FE semi-discretization of our 1D parabolic model problem $\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t)$ for all $(x,t) \in (0,1) \times (0,T)$, with $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(1,t) = g_1(t)$ for all $t \in [0,T]$.

Derive an upper bound for $\|M_h^{-1}K_h\|_{M_h}$ where $\|\cdot\|_{M_h}$ is the operator norm w.r.t. $\|\cdot\|_{M_h}$. This will imply that the right hand side of problem (8) from the lecture satisfies the assumptions of the Picard-Lindelöf Theorem, and so (8) is uniquely solvable.

Hint: $M_h^{-1}K_h$ is positive definite with respect to the inner product $(\underline{v}_h,\, \underline{w}_h)_{M_h} := (M_h\, \underline{v}_h,\, \underline{w}_h)_{\ell^2}$. Use the Rayleigh quotient and the eigenvalue bounds from Chapter 1 to get an upper bound for $\lambda_{\max}(M_h^{-1}K_h)$.

57 Show that

$$\frac{d}{dt}\|\theta_h(t)\|_H \;\le\; \|\rho_h'(t)\|_H - \frac{\mu_1}{c^2}\,\|\theta_h(t)\|_H \qquad \forall t \in (0,\,T) \text{ a.e.}$$

Hint: See (and modify) the proof of Lemma 2.16.

Then, show that

$$\|\theta_h(t)\|_H \;\le\; e^{-\mu_1 c^{-2} t}\,\|\theta_h(0)\|_H + \int_0^t e^{-\mu_1 c^{-2}(t-s)}\,\|\rho_h'(s)\|_H \, ds.$$

This leads also to an improved estimate in Lemma 2.17, in which the initial values lose influence.

Hint: Estimate the term

$$\frac{d}{ds}\Big[e^{\mu_1 c^{-2} s}\,\|\theta_h(s)\|_H\Big]$$

using the previous result, and integrate over $[0,t]$ w.r.t. s.