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40 Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Show that for $M_\tau = I - \tau A$, $\tau \in \mathbb{R}$:

$$\|M_\tau\|_{\ell_2} \;=\; \max_{\lambda\in\sigma(A)} |1-\tau\,\lambda| \;=\; q(\tau)$$

with $q(\tau) = \max\big(|1-\tau\,\lambda_{\max}(A)|,\, |1-\tau\,\lambda_{\min}(A)|\big)$.

Now, assume additionally that $A \in \mathbb{R}^{n\times n}$ is indefinite with at least one positive and one negative eigenvalue. For this setup, use the previous result to show

$$\|M_\tau\|_{\ell_2} \;>\; 1 \qquad \forall \tau \neq 0.$$

41 Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let $\lambda_- < 0$ and $\lambda_+ > 0$ be two eigenvalues with their corresponding eigenvectors e_- and e_+. Show that there is *no* parameter $\tau \in \mathbb{R}$ such that Richardson's method

$$x_{k+1} \;=\; x_k + \tau(b - A\,x_k)$$

converges for the initial vector $x_0 = x + e_- + e_+$, where $x = A^{-1}\,b$.

42 Let A be self-adjoint and positive definite with respect to $(\cdot,\cdot)$, and $b \in \mathbb{R}^n$. Recall the energy functional

$$J_A(y) := \tfrac{1}{2}(A\,y,\, y) - (b,\, y).$$

Let $x_k \in \mathbb{R}^n$ be a given approximation and $p_k \in \mathbb{R}^n$ a given search direction. Compute

$$\alpha_k \;=\; \operatorname*{argmin}_{\alpha\in\mathbb{R}} J_A(x_k + \alpha\,p_k),$$

in terms of p_k, r_k, and A (where $r_k := b - A\,x_k$).

43 (a) Consider the search direction $p_k = r_k + \beta_{k-1}p_{k-1}$ of the Conjugate Gradient method. Compute β_{k-1} such that $(p_{k-1}, p_k)_A = 0$.

(b) For α_k defined as in the lecture (or as in Example 42 in the Tutorial), show that $(r_k, p_{k-1}) = 0$.

44 Show Lemma 1.56 (c) and (d), i.e.

$$\forall j = 0,\ldots,k-1:\;\; (r_k, p_j) = 0 \quad\text{and}\quad (p_k, p_j)_A = 0$$

Hint: Use one Induction over k to prove both statements. Additionally, you may use that $\forall j < k:\;\; (r_{k+1}, Ap_j) = 0$ since $\forall j < k:\;\; Ap_j \in A(\mathcal{K}_k(A, r_0)) \subset \mathcal{K}_{k+1}(A, r_0)$

Programming.

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45 Solve a boundary value problem of your choice with the Conjugate Gradient method. Use `cg.hh` from the webpage and perform the necessary modifications.

Report the number of iterations for 2^k elements, $k = 4,\ldots,10$. Compare to the number of iterations needed by Richardson's method (for the mesh-sizes treated also in Exercise 39) and the theoretical results from the lecture.