34 Let $\mathcal{T}_h$ be an equidistant mesh of $(0,1)$, let V_{0h} be the space of continuous piecewise affine linear functions that vanish at 0, and let K_h be the stiffness matrix corresponding to the 1D model problem. Show that there exists a constant $C > 0$ independent of h such that

$$\kappa(K_h) \;\geq\; C\, h^{-2}$$

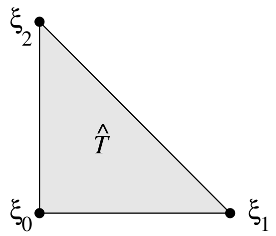
Hint:
Use the Rayleigh quotient for the *special* vector $\underline{v}_h = (1, 0, \ldots, 0)^\top$ to obtain a lower bound for $\lambda_{\max}(K_h)$.
For an upper bound of $\lambda_{\min}(K_h)$, use the vector $\underline{v}_h = (h, 2h, 3h, \ldots, 1)^\top$.

35 Let $\widehat{T} := \{(x,\, y) \in \mathbb{R}^2 : x \geq 0,\; y \geq 0,\; x + y \leq 1\}$ denote the two-dimensional reference element with the corner points $\xi_0 = (0,\, 0)$, $\xi_1 = (1,\, 0)$, and $\xi_2 = (0,\, 1)$. Let $\widehat{\varphi}_0$, $\widehat{\varphi}_1$, and $\widehat{\varphi}_2$ denote affine linear functions on $\widehat{T}$ that fulfill

$$\widehat{\varphi}_i(\xi_j) = \delta_{ij} \qquad \forall i,\, j \in \{0,\, 1,\, 2\}.$$

Derive an explicit formula for $\widehat{\varphi}_0$, $\widehat{\varphi}_1$, and $\widehat{\varphi}_2$ in terms of $\xi = (\xi^{(1)},\, \xi^{(2)})$.

Compute the corresponding element stiffness matrix of the two-dimensional model problem $(-\Delta u = f)$ for a triangle $T = \widehat{T}$ (in this case, the reference triangle is part of the mesh).



36 Consider the function

$$u(x,y) = \sqrt[4]{-\log\sqrt{x^2+y^2}}$$

on the two-dimensional domain

$$\Omega = \{(x,y) \in \mathbb{R}^2 \;:\; x^2 + y^2 < 0.5\}\ .$$

Show $u \in H^1(\Omega)$. Is u also in $C(\bar{\Omega})$?
Hint: Compute $\|u\|^2_{L^2(\Omega)}$ and $|u|^2_{H^1(\Omega)}$ by transforming to polar coordinates, and show $\|u\|_{H^1(\Omega)} < \infty$.

Programming.

37 – If your student ID (Matrikelnummer) is *even* (dividable by 2) consider the boundary value problem

$$-u''(x) \;=\; 2\,x+1, \qquad u(0)=3, \qquad u'(1)=-1/2.$$

Find the *exact* solution u of this equation – it must be a cubic polynomial!

– If your student ID is *odd* (not dividable by 2) consider the exact solution

$$u(x) \;=\; \sin\left(\frac{5\,\pi}{2}\left(x-\frac{1}{5}\right)\right),$$

calculate $f(x)=-u''(x)$, $g_0=u(0)$, and $g_1=u'(1)$ and solve the corresponding boundary value problem numerically.

Implement the exact solution as a function `double mySolution (double x);` and use `ApproxL2Error` to approximate the errors $\|u-u_h\|_{L^2(0,1)}$ for a series of equidistant meshes with 32, 64, 128, 256, 512, and 1024 elements. Report h and the error. How do the results relate to the theory you heard in the lecture?

38 Implement a function

```
void SMatrix :: Mult (const Vector& v, Vector& r) const;
```

in your matrix class, which computes `r`$=K_h\,v$ for $v=$`v`.

39 Solve a boundary value problem of your choice with Richardson's iteration instead of the Thomas algorithm (exercise 31). Use `richardson.hh` from the webpage and perform the necessary modifications.

How do you have to choose τ in order to make the iteration converge?

Report the number of iterations for 2, 4, 8, 16, 32, and 64 elements. How do the results relate to the theory you heard in the lecture?