28 Prove an improved version of Céa‘s Lemma for the symmetric case:

Let the assumptions of Céa's Lemma (see lecture) hold. Additionally, let $b(\cdot,\cdot)$ be symmetric. Then

$$\|u - u_h\|_W \le \sqrt{\frac{\mu_2}{\mu_1}} \inf_{w_h \in W_h} \|u - w_h\|_W \,. \tag{6.1}$$

Hint: Work in the *energy norm* $\|v\|_B := \sqrt{b(v,v)}$ and apply Céa. Then estimate the norms in order to get 6.1.

29 Construct the efficient Gauss algorithm which exploits the tridiagonal matrix structure of the $n \times n$ system of linear equations $Ax = b$, where

$$A = \begin{pmatrix} d_1 & c_1 & & 0 \\ a_1 & d_2 & c_2 & \\ & a_2 & d_3 & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix}.$$

Step 1: Eliminate lower off-diagonal:

Show that $Ax = b$ is equivalent to a system $\tilde{A}x = \tilde{b}$, where $\tilde{A}$ and $\tilde{b}$ have the structure

$$\tilde{A} = \begin{pmatrix} 1 & \tilde{c}_1 & & 0 \\ & 1 & \tilde{c}_2 & \\ & & 1 & \ddots \\ 0 & & & \ddots \end{pmatrix} \qquad \text{and} \qquad \tilde{b} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \end{pmatrix}.$$

For all i, starting at $i = 1$, provide the expressions $\tilde{c}_i, \tilde{b}_i$ in terms of a_i, b_i, c_i and the previously computed coefficients $\tilde{c}_{i-1}, \tilde{b}_{i-1}$, as appropriate.

Step 2: Solve the system $\tilde{A}x = \tilde{b}$:

For all i, now starting at $i = n$, determine x_i from $\tilde{b}_i, \tilde{c}_i$ and x_{i+1}, as appropriate.

30 Derive the variational formulation of the d-dimensional model problem: $\Omega \subset \mathbb{R}^d$ Lipschitz domain, $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$, find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{aligned} -\operatorname{div}\big(A(x)\,\nabla u(x)\big) + \vec{b}(x)\cdot\nabla u(x) + c(x)\,u(x) &= f(x) && \forall x \in \Omega, \\ u(x) &= g_D(x) && \forall x \in \Gamma_D\,, \\ \big(A(x)\,\nabla u(x)\big)\cdot\vec{n}(x) &= g_N(x) && \forall x \in \Gamma_N\,, \end{aligned}$$

where the scalar functions f, g_D, g_N, and the coefficients $A(x) \in \mathbb{R}^{d\times d}$, $\vec{b}(x) \in \mathbb{R}^d$, and $c(x) \in \mathbb{R}$ are given.

31 Consider the variational formulation of the d-dimensional model problem. Assume that $\text{meas}_{d-1}(\Gamma_D) > 0$, that $f \in L^2(\Omega)$, $g_N \in L^2(\Gamma_N)$, and $g_D \in H^{1/2}(\Gamma_D)$, which means that $g_D \in L^2(\Gamma_D)$ and there exists $g \in H^1(\Omega) : \gamma_D g = g_D$. Show that $a(\cdot,\cdot)$ and $\langle F, \cdot\rangle$ are $H^1(\Omega)$-bounded and that $a(\cdot,\cdot)$ is V_0-coercive.

Programming.

32 Solve the system $K_h \underline{u}_h = \underline{f}_h$ in optimal complexity using Gaussian elimination exploiting the tridiagonal structure.
Hint: See exercise 29!

Use this algorithm to solve the boundary value problem

$$\begin{aligned} -u''(x) &= f(x), \\ u(0) &= g_0\,, \\ u'(1) &= g_1\,, \end{aligned}$$

with $f(x) = 3\,x + 1$, $g_0 = 3$, and $g_1 = -0.5$.

Try to plot your solution. If you are unsure how to do this, feel inspired by the instructions on the webpage.

33 Write a function

```
double ApproxL2Error (const Mesh& mesh, const Vector& uh,
                      RealFunction sol);
```

which approximates the error $\|u - u_h\|_{L^2(0,1)}$ (where uh=$\underline{u}_h$ and sol=u) using the *midpoint rule* on each element:

$$\|u - u_h\|^2_{L^2(0,1)} = \sum_{k=1}^{n_h} \int_{T_k} |u - u_h|^2\,dx \approx \sum_{k=1}^{n_h} h_k\,|u(x_k^*) - u_h(x_k^*)|^2\,,$$

where $x_k^* := \frac{1}{2}(x_{k-1} + x_k)$ is the midpoint of element T_k and $h_k = x_k - x_{k-1}$.