In the lecture, the coercivity of the bilinear form

$$a(w,\,v) \;=\; \int_0^1 \big[a(x)\,w'(x)\,v'(x) + b(x)\,w'(x)\,v(x) + c(x)\,w(x)\,v(x)\,\big]\,dx\,, \qquad (3.1)$$

on the space $V_0 = \{v \in H^1(0,\,1) : v(0) = 0\}$ has been shown for the special case $a \equiv 1$, $b \equiv 0$, $c \equiv 0$. In the following three exercises, we consider more general situations. In all situations, you will need the estimate

$$a(v,\,v) \;\geq\; a_0\,|v|^2_{H^1(0,\,1)} + \int_0^1 b(x)\,v'(x)\,v(x)\,dx + c_0\,\|v\|^2_{L_2(0,\,1)}\,, \qquad (3.2)$$

where $a_0 := \inf_{x\in(0,\,1)} a(x)$ and $c_0 := \inf_{x\in(0,\,1)} c(x)$.

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10 Show the coercivity of $a(\cdot,\cdot)$ on $V_0 = \{v \in H^1(0,\,1) : v(0) = 0\}$ under the assumptions

$$a_0 > 0\,, \qquad C_F\,\|b\|_{L_\infty(0,\,1)} < a_0\,, \qquad c_0 \geq 0\,,$$

where C_F is the constant in Friedrichs' inequality.
Hint: Use Cauchy's inequality to show the estimate

$$\int_0^1 b(x)\,v'(x)\,v(x)\,dx \;\geq\; -\|b\|_{L_\infty(0,\,1)}\,|v|_{H^1(0,\,1)}\,\|v\|_{L_2(0,\,1)}$$

and use it in (3.2).

11 Show the coercivity of $a(\cdot,\cdot)$ on the whole space $H^1(0,\,1)$ under the assumptions

$$a_0 > 0\,, \qquad \|b\|_{L_\infty(0,\,1)} < 2\sqrt{a_0\,c_0}\,, \qquad c_0 > 0\,.$$

Hint: Show that

$$a(v,\,v) \;\geq\; q\big(|v|_{H^1(0,\,1)},\,\|v\|_{L_2(0,\,1)}\big)\,,$$

with

$$q(\xi,\,\eta) \;:=\; a_0\,\xi^2 - \|b\|_{L_\infty(0,\,1)}\,\xi\,\eta + c_0\,\eta^2\,.$$

Finally, show and use that

$$q(\xi,\,\eta) \;\geq\; a_0\,C\,\xi^2\,, \quad \text{and} \quad q(\xi,\,\eta) \;\geq\; c_0\,C\,\eta^2\,,$$

with $C = 1 - \frac{\|b\|^2_{L_\infty(0,\,1)}}{4\,a_0\,c_0}$.

12 Show the coercivity of $a(w,\,v)$ on the space $V_0 = \{v \in H^1(0,\,1) : v(0) = 0$ under the assumptions

$$a_0 > 0\,, \qquad b(x) = b_0 \geq 0\,, \qquad c_0 \geq 0\,,$$

where b_0 is a constant.
Hint: Show and use that

$$\int_0^1 v'(x)\,v(x)\,dx \;=\; \frac{1}{2}\,v(x)^2\Big|_0^1 \;\geq\; 0 \qquad \forall v \in V_0\,.$$

13 Prove Lemma 1.29 from the lecture: Let V be a Hilbert space, $a(\cdot,\,\cdot) : V \times V \to \mathbb{R}$ a bilinear form that is symmetric and non-negative on V. Let $V_0 \subset V$ and $V_g = g + V_0$ with $g \in V$. Then

$$\text{find } u \in V_g : \quad a(u,\,v) = \langle F,\,v\rangle \quad \forall v \in V_0$$

is equivalent to

$$\text{find } u \in V_g : \quad J_a(u) = \inf_{v \in V_g} J_a(v),$$

with $J_a(v) = \frac{1}{2}a(v,\,v) - \langle F,\,v\rangle$.
Hint: Follow the proof of Lemma 1.24.

14 Consider our 1D model problem from the lecture with $g_0 = 0$, i.e.

$$\text{Find } u \in V_0 : \quad \underbrace{\int_0^1 u'\,v'\,dx}_{=:\langle Au,v\rangle} \;=\; \underbrace{\int_0^1 f\,v\,dx + g_1\,v(1)}_{=:\langle F,v\rangle} \qquad \forall v \in V_0\,,$$

where $V_0 = \{v \in H^1(0,1) : v(0) = 0\}$.

Specify precisely the boundary value problem for $w_k \in V_0$ which corresponds to

$$w_k \;=\; \mathcal{R}(F - A\,u_k),$$

where $\mathcal{R} : V_0^* \to V_0$ is the Riesz isomorphism.
Hint: Follow the steps at the end of Section 1.1 in the lecture notes ("integration by parts backwards")

15 Consider the system of linear equations,

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}}_{=:\underline{u}} = \underbrace{\begin{pmatrix} ? \\ f_2 \\ f_3 \end{pmatrix}}_{=:\underline{f}}.$$

with the side-condition $u_1 = 3$. There are two approaches to the solution:

(1) *Direct approach:*
Substitute u_1 by the given value in the 2^{nd} and 3^{rd} equation, and reduce the system to a 2-by-2 system for u_2 and u_3.

(2) *Variational approach:*
Find $\underline{u} \in \mathbb{R}^3, u_1 = 3$ such that

$$\underbrace{\underline{v}^T A\,\underline{u}}_{=:a(\underline{u},\underline{v})} = \underbrace{\underline{v}^T\,\underline{f}}_{=:f(\underline{v})} \qquad \forall \underline{v} \in \mathbb{R}^3, v_1 = 0.$$

Is (2) a variational formulation? Show (1) $\Leftrightarrow$ (2).

Then, consider the following questions:

- Can the condition $v_1 = 0$ be dropped?
- Can more conditions on $\underline{v}$ be introduced?