$\boxed{04}$ Consider the piecewise constant coefficient function $a \in L^\infty(0,\,1)$,

$$a(x) = \begin{cases} a_1 & \text{for } x \in \left[0,\, \frac{1}{2}\right], \\ a_2 & \text{for } x \in \left(\frac{1}{2},\, 1\right], \end{cases}$$

with positive constants $a_1 \neq a_2$. Derive a variational formulation for the boundary value problem

$$\begin{aligned} -a(x)\, u''(x) &= f(x) \qquad \forall x \in (0,\,1) \setminus \{\tfrac{1}{2}\}, \\ u(0) &= g_1\,, \qquad u(1) \;=\; g_2\,, \end{aligned}$$

together with the *transmission conditions*

$$u(\tfrac{1}{2}^-) \;=\; u(\tfrac{1}{2}^+)\,, \qquad a_1\, u'(\tfrac{1}{2}^-) \;=\; a_2\, u'(\tfrac{1}{2}^+)\,,$$

where $w(\frac{1}{2}^-)$ and $w(\frac{1}{2}^+)$ denote the left sided and right sided limit of a function $w$, respectively.
*Hint:* Integration by parts is only valid on subintervals!

$\boxed{05}$ Let the sequence $(u_k)_{k\in\mathbb{N}}$ of functions be defined by

$$u_k(x) = \begin{cases} 2x & \text{for } x \in \left[0,\, \frac{1}{2} - \frac{1}{2k}\right], \\ 1 - \frac{1}{2k} - 2k\left(x - \frac{1}{2}\right)^2 & \text{for } x \in \left(\frac{1}{2} - \frac{1}{2k},\, \frac{1}{2} + \frac{1}{2k}\right), \\ 2(1-x) & \text{for } x \in \left[\frac{1}{2} + \frac{1}{2k},\, 1\right]. \end{cases}$$

Show that $u_k \in C^1[0,\,1]$. Let $u$ be defined by

$$u(x) = \begin{cases} 2x & \text{for } x \in \left[0,\, \frac{1}{2}\right], \\ 2(1-x) & \text{for } x \in \left(\frac{1}{2},\, 1\right]. \end{cases}$$

Find out if $u,\, u_k \in H^1(0,\,1)$ or not and justify your answer. Calculate $\|u_k - u\|_{H^1(0,\,1)}$ (maybe with a little help from Mathematica/Maple) or find a suitable bound for it in order to show that

$$\lim_{k\to\infty} \|u_k - u\|_{H^1(0,\,1)} = 0\,.$$

Use these results to show that $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $C^1[0,\,1]$ with respect to the $H^1$-norm, but that there exists no limit in $C^1[0,\,1]$.

$\boxed{06}$ Show *Poincaré's inequality*: There exists a constant $C_P > 0$ such that

$$\|v\|_{L_2(0,\,1)} \;\le\; C_P \left\{ |v|^2_{H^1(0,\,1)} + \left( \int_0^1 v(x)\, dx \right)^2 \right\}^{1/2} \qquad \forall v \in H^1(0,\,1)\,.$$

*Hint:* Integrate the identity

$$v(y) \;=\; v(x) + \int_x^y v'(z)\, dz$$

over the whole interval $(0,\,1)$ with respect to $x$. The rest of the proof is then similar to the one of Friedrichs' inequality (see your lecture notes).

$\boxed{07}$ Derive the variational formulation

$$\text{find } u \in V_g : \quad a(u,\, v) = \langle F,\, v\rangle \qquad \forall v \in V_0 \tag{2.1}$$

of the *pure Neumann boundary value problem*

$$\begin{aligned} -u''(x) &= f(x) \qquad \text{for } x \in (0,\,1)\,, \\ -u'(0) &= g_0\,, \\ u'(1) &= g_1\,, \end{aligned}$$

and show the following statements:

(a) If (2.1) has a solution, then

$$\langle F,\, c\rangle = 0, \qquad \forall c \in \mathbb{R}\,. \tag{2.2}$$

(b) If $u$ is a solution of (2.1), then, for any constant $c \in \mathbb{R}$, $\widehat{u} := u + c$ is also a solution.

(c) If we choose $c = -\int_0^1 u(x)\,dx$, then

$$\widehat{u} \in \widehat{V} = \left\{ v \in H^1(0,\,1) : \int_0^1 v(x)\,dx = 0 \right\}$$

(d) If $\widehat{u} \in \widehat{V}$ solves the variational problem

$$a(\widehat{u},\, \widehat{v}) = \langle F,\, \widehat{v}\rangle \quad \forall v \in \widehat{V}\,,$$

and if the condition (2.2) holds, then $\widehat{u}$ solves also (2.1).
*Hint:* Each test function $v \in H^1(0,\,1)$ can be written as $v(x) = \widehat{v}(x) + \overline{v}$ with $\overline{v} = \int_0^1 v(x)\,dx$ and $\widehat{v} \in \widehat{V}$.

$\boxed{08}$ *The pure Neumann problem (continuation of exercise* $\boxed{06}$*).*
Show that the weak formulation of the pure Neumann problem has a solution if and only if $\forall c \in \mathbb{R} : \langle F, c\rangle = 0$, and that the solution is unique up to an additive constant.
*Hint:* Use Poincaré's inequality to show the $\widehat{V}$-coercivity of $a(\cdot,\cdot)$.

$\boxed{09}$ *A Robin problem.* Consider the variational formulation of

$$\begin{aligned} -u''(x) &= f(x) \qquad \text{for } x \in (0,\,1), \\ -u'(0) &= g_0 - \alpha_0\, u(0) \\ u'(1) &= g_1 \end{aligned}$$

with appropriate choices of $V_0$ and $V_g$. Show that if $\alpha_0 > 0$, then the corresponding bilinear form is $V_0$-coercive.
*Hint:* convince yourself that $\frac{1}{2}\,\|v\|^2_{L^2(0,1)} \le \|v - v(0)\|^2_{L^2(0,1)} + |v(0)|^2$ and use Friedrichs' inequality to bound the first summand.