As it was discussed in the lecture, the subspace correction preconditioner $\widehat{B}_k$ satisfies, for any $k \leq l$, the following identity (XZ identity):

$$\widehat{\mathbf{v}}_k^T \widehat{B}_k \widehat{\mathbf{v}}_k = \min_{\widehat{\mathbf{v}}_k = \widetilde{P}_k \mathbf{v}_k + \ldots + \widetilde{P}_l \mathbf{v}_l} \left(\mathbf{v}_l^T \overline{M}_l \, \mathbf{v}_l + \sum_{j=k}^{l-1} \| M_j^T \mathbf{v}_j + \widetilde{P}_j^T A \, \widehat{\mathbf{v}}_{j+1} \|^2_{(M_j^T + M_j - A_j)^{-1}} \right). \quad (8.1)$$

$\boxed{19}$ The XZ identity is traditionally formulated in terms of the operators

$$T_k = \widetilde{P}_k M_k^{-1} \widetilde{P}_k^T A, \quad T_k^* = \widetilde{P}_k M_k^{-T} \widetilde{P}_k^T A \quad \text{and} \quad \overline{T}_k = \widetilde{P}_k \overline{M}_k^{-1} \widetilde{P}_k^T A.$$

Show that the operators $\overline{T}_k$ are invertible on $\overline{\boldsymbol{V}}_k$.

$\boxed{20}$ Prove the following formulation of the XZ identity:

$$\widehat{\mathbf{v}}_k^T \widehat{B}_k \widehat{\mathbf{v}}_k = \min_{\widehat{\mathbf{v}}_k = \widetilde{P}_k \mathbf{v}_k + \ldots + \widetilde{P}_l \mathbf{v}_l} \Bigg(\left(\overline{T}_l^{-1} \widetilde{P}_l \, \mathbf{v}_l, \widetilde{P}_l \, \mathbf{v}_l \right)_A + \sum_{j=k}^{l-1} \left(\overline{T}_j^{-1} \left(\widetilde{P}_j \, \mathbf{v}_j + T_j^* \, \widehat{\mathbf{v}}_{j+1} \right), \left(\widetilde{P}_j \, \mathbf{v}_j + T_j^* \, \widehat{\mathbf{v}}_{j+1} \right) \right)_A \Bigg)$$

with the operators from Exercise 19 and the notation $(\mathbf{u}, \mathbf{v})_A = \mathbf{v}^T A \, \mathbf{u}$.