17 Consider Poisson's equation on the unit square $\Omega := (0,1) \times (0,1)$, i.e.

$$-\triangle u = f \qquad \text{in } \Omega, \tag{7.1}$$
$$u = 0 \qquad \text{on } \partial\Omega, \tag{7.2}$$

and the corresponding variational problem, i.e., find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \, \nabla v \, dx = \int_\Omega f \, v \, dx \quad \forall v \in H_0^1(\Omega). \tag{7.3}$$

Use the command `A = gallery('poisson',n)` in MATLAB to generate the block tridiagonal sparse stiffness matrix $A = A^{(k)}$ resulting from discretizing the bilinear form in problem (7.3) with the 5-point operator on a uniform mesh. The parameter `n` should be of the form $n_k + 1$ with $n_k = 2^k$ at a level k. The produced mesh $\mathcal{T}_k$ has a lexicographical node numbering. Hence the ordering of unknowns in $A^{(k)}$ has a lexicographical ordering as well.

The mesh $\mathcal{T}_k$ at level k and the coarser mesh $\mathcal{T}_{k-1}$ at level $k-1$ are nested, in other words, the nodes of the coarser mesh are also nodes of the finer mesh.

Transform the matrix $A^{(k)}$ into a 2×2 block form

$$\tilde{A}^{(k)} = \begin{bmatrix} \tilde{A}_{11}^{(k)} & \tilde{A}_{12}^{(k)} \\ \tilde{A}_{21}^{(k)} & \tilde{A}_{22}^{(k)} \end{bmatrix},$$

where the block $\tilde{A}_{22}^{(k)}$ corresponds to those nodes of the fine mesh which are also present on the coarser mesh and the block $\tilde{A}_{11}^{(k)}$ corresponds to the remaining nodes. Both blocks have a lexicographical ordering.

Hint: The transformation can be formally performed by a permutation of rows and columns of the level-k stiffness matrix, i.e.

$$\tilde{A}^{(k)} = P^T A^{(k)} P.$$

18 Construct the (two-level) hierarchical basis representation

$$\hat{A}^{(k)} = \begin{bmatrix} \hat{A}_{11}^{(k)} & \hat{A}_{12}^{(k)} \\ \hat{A}_{21}^{(k)} & \hat{A}_{22}^{(k)} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11}^{(k)} & \hat{A}_{12}^{(k)} \\ \hat{A}_{21}^{(k)} & A^{(k-1)} \end{bmatrix}$$

of the level-k stiffness matrix at an arbitrary level k.

Hint: As we know from the lecture, it is possible to compute $\hat{A}^{(k)}$ from the relation

$$\hat{A}^{(k)} = (J^{(k)})^T \tilde{A}^{(k)} J^{(k)},$$

where the transformation matrix $J^{(k)}$ at level k has the form

$$J^{(k)} = \begin{bmatrix} I_1 & J_{12}^{(k)} \\ 0 & I_2 \end{bmatrix}$$

and relates the nodal point vectors for the hierarchical and the standard basis in the following way:

$$\mathbf{v} := \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = J^{(k)} \begin{bmatrix} \hat{\mathbf{v}}_1 \\ \hat{\mathbf{v}}_2 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{v}}_1 + J_{12}^{(k)} \hat{\mathbf{v}}_2 \\ \hat{\mathbf{v}}_2 \end{bmatrix}.$$

Moreover, in case of linear shape functions and triangular elements the submatrix $J_{12}^{(k)}$ has a very simple form: it has exactly two nonzero entries in each row. All these nonzeros are equal to $1/2$ and they occur in those positions that correspond to connections between the fine nodes (newly added nodes) and the coarse nodes (associated with $A^{(k-1)}$).

Remark: Exploit the sparsity to compute the triple matrix product $(J^{(k)})^T \tilde{A}^{(k)} J^{(k)}$, which requires only $\mathcal{O}(N_k)$ arithmetic operations, where $N_k = (n_k + 1)(n_k + 1)$ is the number of grid points.