15 Prove the following Lemma:

Lemma 6.1 *For all* $(\alpha, \beta) \in D$*, where*

$$D := \Big\{(\alpha,\beta) \in \mathbb{R}^2 : -\frac{1}{2} < \alpha \leq 1, \max\{-\frac{\alpha}{\alpha+1}, |\alpha|\} \leq \beta \leq 1\Big\},$$

there holds the inequality

$$\frac{\alpha\beta + \alpha + \beta + 1}{(\alpha+\beta+1)(\alpha+\beta+2)} > \frac{4}{15}.$$

16 Consider the GCG (generalized conjugate gradient) method and show that if

$$0 \leq m_n \leq m_{n-1} + 1 \qquad \forall n = 1, 2, ...,$$

where $\{m_n\}_{n=1,2,...}$ is some given sequence of integer parameters, then

$$\langle \boldsymbol{p}_{(k)}, A\boldsymbol{p}_{(j)} \rangle = 0 \qquad \forall j, k \text{ such that } n - m_n \leq j < k < n, \tag{6.1}$$

$$\langle \boldsymbol{r}_{(k)}, \boldsymbol{p}_{(j)} \rangle = 0 \qquad \forall j, k \text{ such that } n - m_n \leq j < k \leq n \tag{6.2}$$

and that the following optimality property holds:

$$\|\boldsymbol{x} - \boldsymbol{x}_{(n)}\|_A = \min_{\boldsymbol{p} \in \operatorname{span}\{\boldsymbol{p}_{(n-m_n)}, \ldots, \boldsymbol{p}_{(n-1)}\}} \|\boldsymbol{x} - \boldsymbol{x}_{(n-m_n)} - \boldsymbol{p}\|_A. \tag{6.3}$$