13 Consider the block tridiagonal sparse matrix A of order $N = n^2$ resulting from discretizing Poisson's equation with the 5-point operator on an n-by-n uniform mesh assuming (homogeneous) Dirichlet boundary conditions all over the boundary. In MATLAB, this matrix can be generated with the following command:

`A = gallery('poisson',n).`

(a) Compute the condition number $\kappa(A)$ of A for $n = 2^k$, where $k = 1, 2, ..., k_{\max}$ and $k_{\max} \in \{6, 7, 8\}$ depending on the available memory on your laptop. Depict $\kappa(A)$ in a diagram with (doubly) logarithmic scale (log-log-plot) for $h^{-1} = n = 2^k$, $k = 1, 2, ..., k_{\max}$. How does $\kappa(A)$ depend on h?

(b) Compute an incomplete LU factorization of A, i.e., $A \approx K_0 = L_0 U_0$ as well as a modified incomplete LU factorization of A, i.e., $A \approx K_1 = L_1 U_1$ both with no (additional) fill ((M)ILU(0)), that means that L_p, U_p for $p = 0, 1$ have their only nonzero entries in positions where A has nonzeros. Study the condition number of $B_0 = K_0^{-1} A$ and $B_1 = K_1^{-1} A$ for $h^{-1} = n = 2^k$, $k = 1, 2, ..., k_{\max}$ and compare its growth rate with that of $\kappa(A)$.

Hint: The commands

```
dAKmax = eigs(A,K,1,'lm',options);
dAKmin = eigs(A,K,1,'sm',options);
```

can be used to compute the largest and smallest eigenvalue of the generalized eigenproblem $A\mathbf{v} = \lambda K \mathbf{v}$, respectively. In the options one can specify the precision.

14 (a) Solve the system of linear equations $A\mathbf{x} = \mathbf{b}$ with A from Exercise 13 and a right-hand side vector $\mathbf{b}$ with random entries, e.g., `b = 100*rand(N,1)/N`, using the preconditioned conjugate gradient (PCG) method for some fixed tolerance for the relative residual norm, e.g., `tol = 1e-5`, for $h^{-1} = n = 2^k$, $k = 1, 2, ..., k_{\max}$ and plot the number of iterations in case of

(i) no preconditioning,
(ii) using the preconditioner K_0 obtained from ILU(0) factorization,
(iii) using the preconditioner K_1 obtained from MILU(0) factorization

in a proper diagram.

(b) How is the number of iterations - needed to satisfy the stopping criterion - related to the condition number (estimates) computed in Exercise 13? What does the theory say?