10 Prove in detail the following convergence estimate for the method of steepest descent, which provides a sharp upper bound:

$$\|\mathbf{e}_{(n)}\|_A^2 \leq \|\mathbf{e}_{(n-1)}\|_A^2 \left(1 - \frac{4}{2 + \kappa(A) + 1/\kappa(A)}\right).$$

Hint: Note from the lecture that

$$\|\mathbf{e}_{(n)}\|_A^2 = \|\mathbf{e}_{(n-1)}\|_A^2 \left(1 - \frac{\langle \mathbf{r}_{(n-1)}, \mathbf{r}_{(n-1)}\rangle^2}{\langle A^{-1}\mathbf{r}_{(n-1)}, \mathbf{r}_{(n-1)}\rangle \langle \mathbf{r}_{(n-1)}, A\mathbf{r}_{(n-1)}\rangle}\right).$$

11 Show that

$$\alpha_n = \frac{\langle \mathbf{r}_{(n-1)}, \mathbf{p}_{(n)}\rangle}{\langle \mathbf{p}_{(n)}, \mathbf{p}_{(n)}\rangle_A} = \frac{\langle \mathbf{r}_{(n-1)}, \mathbf{r}_{(n-1)}\rangle}{\langle \mathbf{p}_{(n)}, \mathbf{p}_{(n)}\rangle_A}$$

minimizes $\phi(\mathbf{x}_{(n-1)} + \alpha_n \mathbf{p}_{(n)})$ with respect to α_n, where $\phi(\mathbf{x}) := \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b}$.

12 Prove the following Lemma:

Lemma 4.1 *Let A be a symmetric positive semidefinite matrix and let $V_1 \times V_2$ be a splitting of the vector space, which is consistent with a two-by-two block form partioning of A. Let $\mathbf{v}_1 \in V_1$ and $\mathbf{v}_2 \in V_2$ and let $\gamma < 1$ satisfy the strengthened Cauchy-Bunyakowski-Schwarz inequality. Then*

(a)

$$\gamma^2 = \sup_{\mathbf{v}_2 \in V_2 \setminus \ker(A_{22})} \frac{\mathbf{v}_2^T A_{21} A_{11}^{-1} A_{12} \mathbf{v}_2}{\mathbf{v}_2^T A_{22} \mathbf{v}_2}.$$

(b) For any $\mathbf{v}_2 \in V_2 \setminus \ker(A_{22})$

$$1 - \gamma^2 \leq \frac{\mathbf{v}_2^T S \mathbf{v}_2}{\mathbf{v}_2^T A_{22} \mathbf{v}_2} < 1,$$

where S stands for the Schur complement. The left-hand side inequality is sharp. The right-hand side inequality is sharp if $\ker(A_{12}) \neq \{0\}$.