06 Consider the coefficients $\tau_{n,k}$ of the matrix polynomial for the first-order Chebyshev iterative method, which are given by

$$\begin{aligned}
\tau_{0,0} &:= 1, \qquad (\tau_{j,-1} := 0 \quad \forall j),\\
\tau_{1,1} &:= \alpha_1,\\
\tau_{1,0} &:= 1-\alpha_1,\\
\tau_{n,n} &:= \alpha_n\, \tau_{n-1,n-1},\\
\tau_{n,k} &:= \alpha_n\, \tau_{n-1,k-1} + (1-\alpha_n)\, \tau_{n-1,k}, \quad n \geq 0, \quad k = 0,1,...,n-1,
\end{aligned}$$

where $(\alpha_i)_{i=1}^n$ is the sequence of relaxation parameters.

Show that $\sum_{k=0}^n \tau_{n,k} = 1$, $n \geq 0$.

07 Prove that the n-th Chebyshev polynomial of the first kind, which is defined recursively via

$$\begin{aligned}
T_0(z) &= 1,\\
T_1(z) &= z,\\
T_{n+1}(z) &= 2\,z\,T_n(z) - T_{n-1}(z), \quad n = 1,2,3,..., \; z \in \mathbb{R},
\end{aligned}$$

has the analytic form

$$T_n(z) = \frac{1}{2}\left(\left(z+\sqrt{z^2-1}\right)^n + \left(z-\sqrt{z^2-1}\right)^n\right), \quad n = 0,1,2,..., \; z \in \mathbb{R}.$$

08 Consider the following second-order method:

$$\begin{aligned}
\mathbf{x}^{(1)} &= \mathbf{x}^{(0)} + \frac{\alpha_0}{2}\, C^{-1}\left(\mathbf{b} - A\mathbf{x}^{(0)}\right), \quad \alpha_0 = \frac{4}{\lambda_1 + \lambda_N}\\
\mathbf{x}^{(n+1)} &= \beta_n\, \mathbf{x}^{(n)} + (1-\beta_n)\, \mathbf{x}^{(n-1)} + \alpha_n\, C^{-1}\left(\mathbf{b} - A\mathbf{x}^{(n)}\right), \quad n \geq 1.
\end{aligned}$$

Show that the algebraic polynomial $P_n(z)$ associated with this polynomial method satisfies the recurrence relation

$$P_{n+1}(z) = (\beta_n - \alpha_n\, z) P_n(z) - (\beta_n - 1) P_{n-1}(z), \quad n \geq 1.$$

09 Prove that for the second-order Chebyshev method (see also Exercise 8), i.e., for $P_n(z) = \tilde{P}_n(z)$, $n \geq 1$, where

$$\tilde{P}_n(z) = \frac{T_n\left(\frac{2z-(\lambda_N+\lambda_1)}{\lambda_N-\lambda_1}\right)}{T_n\left(\frac{-(\lambda_N+\lambda_1)}{\lambda_N-\lambda_1}\right)}$$

is the n-th order Chebyshev polynomial associated with the interval $[\lambda_1, \lambda_N]$, the coefficient sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ can be computed via

$$\beta_n = \frac{\lambda_1+\lambda_N}{2}\,\alpha_n, \quad \alpha_n^{-1} = \frac{\lambda_1+\lambda_N}{2} - \left(\frac{\lambda_N-\lambda_1}{4}\right)^2 \alpha_{n-1}, \quad n \geq 1.$$