Fast Solvers WS 2012/13
Tutorial 1 Thursday, 18 October, S2 054, 9.15-10.00

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01 We have the following weak formulation:

Given $f \in L_2(\Omega)$ find $u \in \mathcal{V} := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, satisfying

$$\mathcal{A}(u, v) = \mathcal{L}(v) \quad \forall v \in \mathcal{V}, \tag{1.1}$$

where Γ_D denotes the Dirichlet boundary and

$$\mathcal{A}(u, v) := \int_\Omega \boldsymbol{a}(\boldsymbol{x}) \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\, d\boldsymbol{x},$$
$$\mathcal{L}(v) := \int_\Omega f(\boldsymbol{x})\, v(\boldsymbol{x})\, d\boldsymbol{x},$$

where the coefficient matrix $\boldsymbol{a}(\boldsymbol{x})$ is assumed to be symmetric positive definite and uniformly bounded in Ω.

Prove that this weak formulation is equivalent to the following abstract minimization problem: Find $u \in \mathcal{V}$ such that

$$\mathcal{F}(u) = \min_{v \in \mathcal{V}} \mathcal{F}(v),$$
$$\mathcal{F}(v) := \frac{1}{2}\mathcal{A}(v, v) - \mathcal{L}(v).$$

02 Let us assume Ω is a polygonal domain and $\mathcal{T}_h$ is a triangulation of Ω with mesh size $h := \max_{T \in \mathcal{T}_h} h_T$ and the mesh is assumed to be quasi-uniform, i.e., for all elements $T \in \mathcal{T}_h$ the conditions $h_T \geq \delta_1 h$ and $\frac{\rho_T}{h_T} \geq \delta_2$ are satisfied. Let $\mathcal{V}_h \subset \mathcal{V}$ be a finite-dimensional subspace of $\mathcal{V}$ with a basis $\Phi = \{\phi_1, \phi_2, ..., \phi_N\}$. We consider the following finite-dimensional problem corresponding to (1.1):

Find $u_h \in \mathcal{V}_h$ such that

$$\mathcal{A}_h(u_h, v_h) = \mathcal{L}(v_h) \quad \forall v_h \in \mathcal{V}_h, \tag{1.2}$$

where

$$\mathcal{A}_h(u_h, v_h) = \mathcal{A}(u_h, v_h) := \int_\Omega \boldsymbol{a}(\boldsymbol{x}) \nabla u_h(\boldsymbol{x}) \cdot \nabla v_h(\boldsymbol{x})\, d\boldsymbol{x},$$
$$\mathcal{L}_h(v_h) = \mathcal{L}(v_h) := \int_\Omega f(\boldsymbol{x})\, v_h(\boldsymbol{x})\, d\boldsymbol{x}.$$

Prove that for all $v_h = \sum_{i=1}^N v_i \phi_i \in \mathcal{V}_h$ with $\mathrm{v} = (v_i) \in \mathbb{R}^N$ we have that

(i) $c_1\, h^2 \|\mathrm{v}\|^2 \leq \|v_h\|^2_{L^2(\Omega)} \leq c_2\, h^2 \|\mathrm{v}\|^2,$

(ii) $\mathcal{A}_h(v_h, v_h) \leq c_3\, h^{-2} \|v_h\|^2_{L^2(\Omega)},$

where the constants c_1, c_2, c_3 only depend on δ_1 and δ_2.