

# T U T O R I A L

## “Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

### **Tutorial 12**

Wednesday, 27 June 2012, Time: 08<sup>30</sup> – 10<sup>00</sup>, Room: T111.

## Programming (continued)

### $L_2$ -error and $H^1$ -error

**59** Write a function

```
double calcElErrorL2 (const Point2D& p0, const Point2D& p1,
                     const Point2D& p2, ScalarField exact,
                     double v0, double v1, double v2);
```

that approximates the element  $L^2$ -error  $\|v - v_h\|_{L^2(\delta_r)}$ , where **exact**= $v$  and  $v_h(x_{\delta_r}(\xi)) = \sum_{\alpha \in A} v^{(r,\alpha)} p^{(\alpha)}(\xi)$  with **v0**= $v^{(r,1)}$  etc.

*Hint:* Use the quadrature rule from Exercise **36** to approximate

$$\|v - v_h\|_{L^2(\delta_r)}^2 = \int_{\delta_r} |v(x) - v_h(x)|^2 dx = \int_{\Delta} |v(x_{\delta_r}(\xi)) - v_h(x_{\delta_r}(\xi))|^2 |\det J_{\delta_r}| d\xi$$

**60** Write a function

```
double calcElErrorH1 (const Point2D& p0, const Point2D& p1,
                     const Point2D& p2,
                     ScalarField Dx1exact, ScalarField Dx2exact,
                     double v0, double v1, double v2);
```

that approximates the element  $H^1$ -error  $|Dv - \nabla v_h|_{L^2(\delta_r)}$ , where  $Dv = \nabla v = (\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2})^T$ , with **Dx1exact**= $\frac{\partial v}{\partial x_1}$ , **Dx2exact**= $\frac{\partial v}{\partial x_2}$  and  $v_h(x_{\delta_r}(\xi)) = \sum_{\alpha \in A} v^{(r,\alpha)} p^{(\alpha)}(\xi)$  with **v0**= $v^{(r,1)}$  etc.

*Hint:* Use the quadrature rule from Exercise **36** to approximate

$$|v - v_h|_{H^1(\delta_r)}^2 = \int_{\delta_r} |Dv(x) - \nabla_x v_h(x)|^2 dx = \int_{\Delta} |Dv(x_{\delta_r}(\xi)) - J_r^{-T} \nabla_{\xi} v_h(x_{\delta_r}(\xi))|^2 |\det J_{\delta_r}| d\xi$$

**61** Write a function

```
double calcErrorL2 (const Mesh& mesh, ScalarField exact,
                    const Vector& solution);
```

that approximates the global  $L^2$ -error  $\|v - v_h\|_{L^2(\Omega)}$ , where `exact=v` and `solution=vh`.

*Hint:* use `calcElErrorL2` in a loop over all elements.

Show that  $u(x_1, x_2) = \frac{1}{4} \cos(2\pi x_1) \cos(4\pi x_2)$  is the unique solution of (1.9) (see Tutorial 08, Exercise **41**). Compute  $\|u - u_h\|_{L^2(\Omega)}$  for each finite element solution  $u_h$  from Exercise **41** for the different meshes.

**62** Write a function

```
double calcErrorH1 (const Mesh& mesh, ScalarField exact,
                    ScalarField Dx1exact, ScalarField Dx2exact,
                    const Vector& solution);
```

that approximates the global  $H^1$ -error  $\|v - v_h\|_{H^1(\Omega)}$ , where `exact=v`, `Dx1exact= $\frac{\partial v}{\partial x_1}$` , `Dx2exact= $\frac{\partial v}{\partial x_2}$`  and `solution=vh`.

*Hint:* use `calcElErrorL2` and `calcElErrorH1` in a loop over all elements.

Compute  $\|u - u_h\|_{H^1(\Omega)}$  for each finite element solution  $u_h$  from Exercise **41** for the different meshes.

## The CHIP-Problem

Recall the CHIP-Problem from the lecture (T08a, T08b, T09)!

**63** Prepare the initial mesh for the CHIP problem as proposed on T09 in your mesh-format, taking care of the appropriate boundary conditions.

*Hint:* If possible use symmetric reduction.

**64** Modify your functions from **35**, **37** and **38**, such that you can assemble the stiffness matrix  $K$  according to the bilinear form

$$a(u, v) = \int_{\Omega} \lambda(x) \nabla u(x) \cdot \nabla v(x) + a(x) u(x) v(x) dx,$$

where  $\lambda(x)$  and  $a(x)$  are given coefficient functions.

**65** Solve the finite element system corresponding to the CHIP problem on T08a with the parameter setting of T08b for the initial mesh of **69**. Solve the same system for uniformly refined meshes with  $h/h_0 = 2, 3, 8, 16$  and visualize the solution.

*Hint:* For incorporating the BC, use the following order: First natural BC, than essential BC.

## A posteriori error estimates

- 66\* Implement the residual error estimator for the CHIP-problem as derived in Exercise 58.
- 67\* Compute the residual error for the CHIP-problem for uniformly refined meshes with  $h/h_0 = 2, 3, 8, 16$  and visualize the error on each element!