T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 11 Tuesday, 26 June 2012, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: S2 / 219.

3.6 Clément's Interpolator

55 Let $\Omega = (0,1)$ and consider the equidistant subdivision into elements $[x_{i-1}, x_i] = [(i-1)h, i\,h]$, $i = 1, \ldots, n$. For each node $x_i = i\,h$, $i = 1, \ldots, n-1$ we define the local L_2-projection $P_i : L_2(x_{i-1}, x_{i+1}) \to \mathcal{P}_0(x_{i-1}, x_{i+1}) = \mathbb{R}$ by

$$\int_{x_{i-1}}^{x_{i+1}} (P_i v)\, q\, dx \;=\; \int_{x_{i-1}}^{x_{i+1}} v\, q\, dx \qquad \forall q \in \mathcal{P}_0(x_{i-1}, x_{i+1}) \quad \forall v \in L_2(x_{i-1}, x_{i+1})\,,$$

where $\mathcal{P}_0(x_{i-1}, x_{i+1})$ are the constant functions on (x_{i-1}, x_{i+1}). Show that

1) $P_i v = \frac{1}{2h} \int_{x_{i-1}}^{x_{i+1}} v(x)\, dx$,
2) $\|v - P_i v\|_{L_2(x_{i-1}, x_{i+1})} \le c\, h\, \|v'\|_{L_2(x_{i-1}, x_{i+1})} \qquad \forall v \in H^1(x_{i-1}, x_{i+1})$.

56 Let $V_0 := H_0^1(0,1)$ and $V_{0h} := \operatorname{span}\{p^{(j)} : j = 1, \ldots, n-1\}$ where $p^{(j)}$ is the nodal basis function associated to the node x_j. We define Clément's interpolator $I_h : L_2(0,1) \to V_{0h} \subset V_0$ by

$$(I_h u)(x) := \sum_{j=1}^{n-1} (P_j u)\, p^{(j)}(x) \qquad \text{for } x \in [0,1]\,.$$

Show that $\quad \|u - I_h u\|_{L_2(0,1)} \;\le\; c\, h\, \|u'\|_{L_2(0,1)} \quad \forall u \in V_0\,.$
Hint: Follow your lecture notes. The difference here is that we have to treat homogeneous Dirichlet boundary conditions ! Show and use the scaled Friedrichs inequality

$$\|u\|_{L_2(x_0, x_1)} \;\le\; c_F\, h\, |u|_{H^1(x_0, x_1)}\,,$$

with $c_F \neq c_F(h)$. The same inequality is valid for the interval (x_{n-1}, x_n).

57 Show that $|u - I_h u|_{H^1(0,1)} \;\le\; c\, \|u\|_{H^1(0,1)}\,.$

3.7 A posteriori error estimates

58 In Section 3.6.2 of our lecture, we derived the residual error estimator for the Dirichlet problem of Poisson's equation. How do we have to modify this estimator such that it works for our model problem CHIP ? Derive the right residual error estimator for the CHIP problem !