

T U T O R I A L

“Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

Tutorial 10

Tuesday, 19 June 2012, Time: 10¹⁵ – 11⁴⁵, Room: S2 / 219.

3.5 Variational Crimes

Consider the one-dimensional BVP to find $u \in V_g = V_0 = H_0^1(0, 1)$:

$$\int_0^1 \lambda(x) u'(x) v'(x) dx = \int_0^1 f(x) v(x) dx \quad \forall v \in V_0, \quad (3.24)$$

where $f \in L^2(0, 1)$ and $\lambda \in L^\infty(0, 1)$. We assume that there exists positive constants $\underline{\lambda}$ and $\bar{\lambda}$ such that $0 < \underline{\lambda} \leq \lambda(x) \leq \bar{\lambda} \quad \forall x \in (0, 1)$ a.e. Let us consider a finite element discretization with continuous linear finite elements ($S(\Delta) = P_1$) on an equidistant grid ($x_i = ih$, $i = \bar{0}, n+1$, $h = 1/(n+1)$, $\delta_r = (x_{r-1}, x_r)$, $r = \bar{1}, n+1$). Now we approximate the bilinear form $a(\cdot, \cdot)$ and the linear form $\langle F, \cdot \rangle$ defined in (3.24) using numerical integration, namely the *midpoint rule*:

$$a_h(u_h, v_h) = \sum_{r=1}^{n+1} h \lambda(x_r^*) u'_h(x_r^*) v'_h(x_r^*) \quad \text{and} \quad \langle F_h, v_h \rangle_h = \sum_{r=1}^{n+1} h f(x_r^*) v_h(x_r^*), \quad (3.25)$$

where $x_r^* = x_{\delta_r}(\frac{1}{2}) = x_{r-1} + \frac{1}{2}h$. To ensure that these expressions are well-defined we assume (for simplicity) that

$$\lambda, f \in W_\infty^1(0, 1).$$

Let $\tilde{u}_h \in V_{0h}$ be such that $a_h(\tilde{u}_h, v_h) = \langle F_h, v_h \rangle_h$ for all $v_h \in V_{0h}$. We are interested whether the error $\|u - \tilde{u}_h\|_{H^1(0,1)}$ obeys the same asymptotics with respect to h than if we compute $a(\cdot, \cdot)$ and $\langle F, \cdot \rangle$ exactly. This investigation will be done using Strang's first lemma. Throughout this tutorial, we choose $\|\cdot\|_{V_0} := |\cdot|_{H^1(0,1)}$ that is a norm in $V_0 = H_0^1(0, 1)$.

- 51** Show that the bilinear and linear forms above fulfill the standard assumptions (33) and the additional assumption (34) from the lecture notes. The latter is called *uniform ellipticity* of the discrete bilinear form $a_h(\cdot, \cdot)$, i.e. there exists a positive constant $\mu_3 \neq \mu_3(h)$ such that

$$a_h(v_h, v_h) \geq \mu_3 \|v_h\|_{V_0}^2 \quad \forall v_h \in V_{0h}.$$

Hint: For the uniform ellipticity, use that $\lambda \geq \underline{\lambda}$ and that the midpoint rule is exact for some (which?) polynomials.

We can now apply Strang's first lemma which yields

$$\begin{aligned} \|u - \tilde{u}_h\|_{V_0} \leq c \left\{ \inf_{v_h \in V_0} \left[\|u - v_h\|_{V_0} + \sup_{w_h \in V_{0h}} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_{V_0}} \right] + \right. \\ \left. + \sup_{w_h \in V_{0h}} \frac{|\langle F, w_h \rangle - \langle F_h, w_h \rangle_h|}{\|w_h\|_{V_0}} \right\} \end{aligned} \quad (3.26)$$

52 For $\varphi \in H^1(\delta_r)$, prove that

$$\left| \int_{\delta_r} \varphi(x) dx - h \varphi(x_r^*) \right| \leq c h |\varphi|_{H^1(\delta_r)},$$

similarly to the exercises in Tutorial 07. Then set $\varphi = f w_h$ and show that

$$|\langle F, w_h \rangle - \langle F_h, w_h \rangle_h| \leq c h \|f\|_{W^{1,\infty}(0,1)} \|w_h\|_{V_0}.$$

53 For $v_h, w_h \in V_{0h}$, prove that

$$|a(v_h, w_h) - a_h(v_h, w_h)| \leq c h |\lambda|_{W_\infty^1(0,1)} \|v_h\|_{V_0} \|w_h\|_{V_0}.$$

Hint: Treat each element separately and use that v'_h, w'_h are constant on each element, so that we are left with $|\int_{\delta_r} \lambda(x) dx - h \lambda(x_r^*)|$. To get an error bound for this term, use Bramble-Hilbert on the reference element.

54 Show that if $u \in H^2(0, 1)$ then

$$\|u - \tilde{u}_h\|_{V_0} \leq c h \left\{ |u|_{H^2(0,1)} + |\lambda|_{W_\infty^1(0,1)} \|u\|_{V_0} + \|f\|_{W_\infty^1(0,1)} \right\}.$$

Hint: Choose $v_h = u_h$ in (3.26), where $u_h \in V_{0h}$ is the finite element solution in the exact case, i.e.

$$a(u_h, v_h) = \langle F, v_h \rangle \quad \forall v_h \in V_{0h}.$$

Show and use that

$$\|u_h\|_{V_0} \leq c(\lambda) \|u\|_{V_0}.$$