T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 07 Thursday, 15 May 2012, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: S2 / 219.

1.4 Generation of systems of Finite Elements Equations

29 Show that the integration rule

$$\int_\Delta f(\xi,\eta)d\xi d\eta \approx \frac{1}{2}\{\alpha_1 f(\xi_1,\eta_1) + \alpha_2 f(\xi_2,\eta_2) + \alpha_3 f(\xi_3,\eta_3)\} \tag{1.5}$$

integrates quadratic polynomials exactly, if the the weights α_i and the integration points (ξ_i,η_i) are choosen as follows: $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$ und $(\xi_1,\eta_1) = (1/2,0)$, $(\xi_2,\eta_2) = (1/2,1/2)$, $(\xi_3,\eta_3) = (0,1/2)$.

Hint: cf. also Exercise 20 !

30 Let us assume that $\mathcal{T}_h = \{\delta_r : r \in \mathbb{R}_h\}$ is a regular triangulation of the polygonally bounded Lipschitz domain $\overline{\Omega} = \cup_{r\in\mathbb{R}_h}\overline{\delta}_r \subset \mathbb{R}^2$ into triangles δ_r, and let $u \in H^2(\Omega)$. Let us now compute the integral

$$I(u) = \int_\Omega u(x)dx$$

by the quadrature rule

$$I_h(u) = \sum_{r\in\mathbb{R}_h} u(x_{\delta_r}(\xi^*))\,|\delta_r|,$$

where $x_{\delta_r}(\cdot)$ maps the unit triangle Δ onto δ_r, and $\xi^* = (1/3,1/3)$. Show that

$$|I(u) - I_h(u)| \le ch^2|u|_{H^2(\Omega)},$$

where c is some generic positive constant. Can you weaken the assumption that $u \in H^2(\Omega)$?

Hint: Use the mapping principle and the Bramble-Hilbert Lemma; cf. also Exercise 20 !

31 Generate the system of finite element equations for the mixed boundary value problem

$$-\Delta u(x_1, x_2) = 1 \quad \forall (x_1, x_2) \in \Omega := (0,1) \times (0,1), \tag{1.6}$$
$$u(x_1, 1) = 0 \quad \forall x_1 \in [0,1], \tag{1.7}$$
$$u(1, x_2) = 0 \quad \forall x_2 \in [0,1], \tag{1.8}$$
$$u_{x_1}(0, x_2) = 1 - x_2 \quad \forall x_2 \in (0,1], \tag{1.9}$$
$$u_{x_2}(x_1, 0) = 1 - x_1 \quad \forall x_1 \in (0,1), \tag{1.10}$$

and for the triangulation shown in the attached figure. Solve this linear system of algebraic equations ! Note that u_{x_1} and u_{x_2} denote the partial derivatives with respect to x_1 and x_2.