T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 02 Tuesday, 27 March 2011, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: S2 219.

1.2 The linear elasticity problem

07 Show that, for the BVP of the first type ($\Gamma_1 = \Gamma$) and for the mixed BVP ($\text{meas}_2(\Gamma_1) > 0$ and $\text{meas}_2(\Gamma_2) > 0$) of the linear elasticity, the following statements hold:

1. $a(.,.)$ is symmetric, i.e., $a(u,v) = a(v,u) \quad \forall u,v \in V$,
2. $a(.,.)$ is nonnegative, i.e., $a(v,v) \geq 0 \quad \forall v \in V$,
3. $a(.,.)$ is positive on $V_0 := \{v \in V = [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_1\}$, if $\text{meas}_2(\Gamma_1) > 0$, i.e., $a(v,v) > 0 \quad \forall v \in V_0 : v \not\equiv 0$.

The equivalence of VF $(9)_{VF}$ and MP $(9)_{MP}$ given in the Lectures then follows from Section. 1.1. of the Lectures.

08 Show that, for the first type ($\Gamma_1 = \Gamma$) BVP of the 3D linear elasticity in the case of an isotrop and homogeneous material, the assumptions of the Lax-Milgram Theorem are fulfiled, i.e. provide constants μ_1 and μ_2 such that:

1) $F \in V_0^*$,

2a) $\exists\, \mu_1 = \text{const} > 0 : \; a(v,v) \geq \mu_1 \parallel v \parallel^2_{H^1(\Omega)} \; \forall v \in V_0$,

2b) $\exists\, \mu_2 = \text{const} > 0 : \; |a(u,v)| \leq \mu_2 \parallel u \parallel^2_{H^1(\Omega)} \parallel v \parallel^2_{H^1(\Omega)} \; \forall u,v \in V_0$.

○ <u>Hint:</u> to the proof of V_0-ellipticity:

+ $a(v,v) \geq 2\mu \int_\Omega \sum_{i,j=1}^3 (\varepsilon_{ij}(v))^2 dx$,
+ Korn's inequality for the BVP of the first type: $V_0 = [H_0^1(\Omega)]^3$, where $H_0^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ auf } \Gamma\}$ (Prove this inequality ! Use integration by parts !),
+ FRIEDRICHS-inequality.

09 Provide the weak form of the iterative method (3) from Section 1.1 of the Lectures

$$u_{n+1} = u_n - \rho(JAu_n - JF) \text{ in } V_0 = \{v \in (H^1(\Omega))^3 : v = 0 \text{ on } \Gamma_1\}, \qquad (1.4)$$

with $n = 0, 1, 2, ...$, and given $u_0 \in V_0$, for the mixed ($u = 0$ on Γ_1 and $\sigma \cdot n = t$ on Γ_2) BVP of the linear elasticity in the case of 3D homogeneous and isotrop material, i.e., derive the weak form (variation formulation) for the calculation of $u_{n+1} \in V_0$! Discuss two cases, in which the norm on V_0 is defined as

$$||u||_{V_0}^2 := \int_\Omega |\nabla u|^2 \, dx \quad \forall u \in V_0, \tag{1.5}$$

or as

$$||u||_{V_0}^2 := \int_\Omega (|\nabla u|^2 + |u|^2) \, dx \quad \forall u \in V_0. \tag{1.6}$$

10* Let us consider the variational formulation:

$$\text{Find } u \in V_g = V_0 : \quad a(u, v) = \langle F, v \rangle \quad \forall v \in V_0 \tag{1.7}$$

of a plane linear elasticity problem in $\Omega = (0, 1) \times (0, 1)$, where

$$\begin{aligned} V_0 = \quad \{ \quad & u = (u_1, u_2) \in V = [H^1(\Omega)]^2 : \\ & u_1 = 0 \text{ on } \Gamma_1 = \{0\} \times [0, 1] \\ & u_2 = 0 \text{ on } \Gamma_2 = [0, 1] \times \{1\}\}, \\ a(u, v) \quad = \quad & \int_\Omega \sum_{i,j,k,l=1}^{2} D_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(v) \, dx = \int_\Omega \sum_{k,l=1}^{2} \sigma_{kl}(u) \varepsilon_{kl}(v) \, dx, \\ \langle F, v \rangle \quad = \quad & \int_\Omega \sum_{i=1}^{2} f_i v_i \, dx + \int_{\Gamma_1} t_2 v_2 \, ds + \int_{\Gamma_2} t_1 v_1 \, ds. \end{aligned}$$

Derive the classical formulation of (1.7) !.