T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 01   Thursday, 20 March 2012, Time: $10^{\underline{15}} - 11^{\underline{45}}$, Room: S2 219.

1 Variational formulation of multi-dimensional elliptic Boundary Value Problems (BVP)

1.1 Scalar Second-order Elliptic BVP

◯ In the lecture (Section 1.2.1) we discussed the BVP (a classical formulation)

$$
\text{Search } u \in X := C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2 \cup \Gamma_3) \cap C(\Omega \cup \Gamma_1): \tag{1.1}
$$

$$
-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u(x) = f(x), x \in \Omega
$$

$+$BC:
- $u(x) = g_1(x), x \in \Gamma_1$
- $\frac{\partial u}{\partial N} := \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial u(x)}{\partial x_j}n_i(x) = g_2(x), x \in \Gamma_2$
- $\frac{\partial u}{\partial N} + \alpha(x)u(x) = \underbrace{g_3(x)}_{\alpha(x)u_A(x)}, x \in \Gamma_3$

and derived the variational formulation

$$
\text{Search } u \in V_g : a(u,v) = <F,v> \quad \forall v \in V_0 \tag{1.2}
$$

with

$$
a(u,v) \quad := \quad \int_{\Omega}\left(\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \sum_{i=1}^{d} b_i\frac{\partial u}{\partial x_i}v + cuv\right)dx + \int_{\Gamma_3}\alpha uv\, ds,
$$

$$
<F,v> \quad := \quad \int_{\Omega} fv\, dx + \int_{\Gamma_2} g_2 v\, ds + \int_{\Gamma_3} g_3 v\, ds
$$

$$
V_g := \{v \in V = W_2^1(\Omega) : v = g_1 \text{ on } \Gamma_1\}
$$

$$
V_0 := \{v \in V : v = 0 \text{ on } \Gamma_1\}
$$

under the assumptions

$$
\left.
\boxed{
\begin{array}{ll}
1) & a_{ij}, b_i, c \in L_\infty(\Omega), \alpha \in L_\infty(\Gamma_3) \\[4pt]
2) & f \in L_2(\Omega), g_i \in L_2(\Gamma_i), i = 2, 3 \\[4pt]
3) & g_1 \in H^{\frac{1}{2}}(\Gamma_1), \text{ i.e., } \exists \tilde{g}_1 \in H^1(\Omega) : \tilde{g}_1|_{\Gamma_1} = g_1. \\[4pt]
4) & \Omega \subset \mathbf{R}^d(\text{bounded}) : \Gamma = \partial\Omega \in C^{0,1} \text{ (Lip boundary)} \\[4pt]
5) & \text{uniform ellipticity:} \\[6pt]
& \left.
\begin{array}{l}
\displaystyle\sum_{i,j=1}^{d} a_{ij}(x)\xi_i; \xi_j \geq \bar{\mu}_1 |\xi|^2 \quad \forall \xi \in \mathbf{R}^d \\[10pt]
a_{ij}(x) = a_{ji}(x) \quad \forall i, j = \overline{1, d}
\end{array}
\right\} \forall \text{ a.e. } x \in \Omega.
\end{array}
}
\right\} \quad (1.3)
$$

01 Formulate the classical assumptions on $\{ a_{ij}, b_i, c, \alpha, f, g_i, \Omega$ resp. $\partial\Omega \}$ for (1.1) !

02 Show that, for sufficiently smooth data, a the generalized solution $u \in V_g \cap X \cap H^2(\Omega)$ of the Boundary Value Problem (2) is also a classical solution, i.e. a solution of (1) !

(1) $\left\{ \begin{array}{l}
\text{Find } u \in X = C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2 \cup \Gamma_3) \cap C(\Omega \cup \Gamma_1) : \\
-\Delta u(x) + u(x) = f(x), x \in \Omega \subset \mathbf{R}^d \text{ (bounded)}, \\
u(x) = g_1(x), \ x \in \Gamma_1, \\
\frac{\partial u}{\partial n}(x) = g_2(x), \ x \in \Gamma_2, \\
\frac{\partial u}{\partial n}(x) = \alpha(x)(g_3(x) - u(x)), \ x \in \Gamma_3
\end{array} \right.$

$?\Downarrow\Uparrow?$

(2) $\left\{ \begin{array}{l}
\text{Find } u \in V_g = \{v \in V = H^1(\Omega) : v = g_1 \text{ on } \Gamma_1\} \text{ such that } \forall v \in V_0 : \\[6pt]
\underbrace{\displaystyle\int_\Omega (\nabla^T u \nabla v + uv)\, dx + \int_{\Gamma_3} \alpha uv\, ds}_{=a(u,v)} = \underbrace{\int_\Omega fv\, dx + \int_{\Gamma_2} g_2 v\, ds + \int_{\Gamma_3} \alpha g_3 v\, ds}_{=<F,v>}
\end{array} \right.$

where $V_0 = \{v \in V = H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}$.

03 Show that the assumptions of the Lax-Milgram-Theorem are satisfied for the variational problem (1.2) under the assumptions (1.3) and the additional assumptions $b_i = 0, c = 0, \alpha(x) \geq \underline{\alpha} = \text{const} > 0 \quad \forall \text{ a.e. } x \in \Gamma_3$ with $\text{meas}_{d-1}(\Gamma_i) > 0, i = 1, 2, 3$!

04 In addition to the assumption (1.3) let us assume that $c(x) \geq \underline{c} = \text{const} > 0 \quad \forall$ a.e. $x \in \Omega, \Gamma_2 = \Gamma_3 = \emptyset$, $g_1 = 0$, and $b_i \not\equiv 0$. Provide conditions for the coefficients $b_i(x)$ of the vector function $b(x) = (b_1(x), ..., b_d(x))^T$ such that the assumptions of Lax-Milgram's Theorem are fulfilled !

○ Hint: For estimating the convection term

$$
\sum_{i=1}^{m} \int_\Omega b_i(x) \frac{\partial u}{\partial x_i}(x) v(x)\, dx = \int_\Omega (b(x), \nabla u(x)) v(x)\, dx,
$$

make use of the identity

$$\int_\Omega (b, \nabla u)v\, dx = -\int_\Omega (b, \nabla v)u\, dx - \int_\Omega u\, v\, \text{div}(b)dx + \int_\Gamma (b, n)\, u\, v\, ds$$

that can easily be proved by partial integration.

05 Derive the variational formulation of the pure Neumann problem for the Poisson equation

$$(*)\left\{\begin{array}{l} -\Delta u(x) = f(x), x \in \Omega \\ \frac{\partial u}{\partial \vec{n}}(x) = 0, x \in \Gamma = \partial\Omega \end{array}\right.,$$

and discuss the question of the existence and the uniqueness of a generalized solution of $(*)$!

○ Hint:
Obviously, $u(x)+c$ with an arbitrary constant $c \in \mathbf{R}^1$ solves $(*)$, if u is the solution of the BVP $(*)$. There are the following ways to analyze the existence of a generalized solution:

1) Set up the variational formulation in $V = H^1(\Omega)$ and apply FREDHOLM-Theory !

2) Set up the variational formulation im factor-space $V = H^1(\Omega)|_{\text{ker}}$ with ker= $\{c : c \in \mathbf{R}^1\} = \mathbf{R}^1$ and apply LAX-MILGRAM-Theorem !

06* Derive the variational formulation of the Dirichlet problem for the Helmholtz equation

$$(**)\left\{\begin{array}{l} -\Delta u(x) - \omega^2 u = f(x), x \in \Omega \\ u(x) = 0, x \in \Gamma = \partial\Omega \end{array}\right..$$

Then discuss the problem of the existence and the uniqueness of the generalized solution of $(**)$! Here ω^2 is a given positive constant.