Traces of $\mathbf{H}(\mathbf{curl})$

Trace operators[1]

$$\begin{aligned}\boldsymbol{\gamma}_T &: \mathbf{C}^\infty(\overline{\Omega}) \to \mathbf{C}^\infty(\partial\Omega) : \boldsymbol{w} \mapsto \boldsymbol{n} \times (\boldsymbol{w} \times \boldsymbol{n}) \\ \boldsymbol{\gamma}_T^\times &: \mathbf{C}^\infty(\overline{\Omega}) \to \mathbf{C}^\infty(\partial\Omega) : \boldsymbol{w} \mapsto \boldsymbol{w} \times \boldsymbol{n}\end{aligned}$$

Theorem 2.26. There exist unique extensions

$$\boldsymbol{\gamma}_T : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\partial\Omega), \qquad \boldsymbol{\gamma}_T^\times : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\partial\Omega)$$

as linear and bounded operators.

WARNING: $\boldsymbol{\gamma}_T$, $\boldsymbol{\gamma}_T^\times$ as stated in Theorem 2.26 are *not surjective!*
First, this is because $(\boldsymbol{a} \times \boldsymbol{n}) \cdot \boldsymbol{n} = 0$ and so

$$\boldsymbol{\gamma}_T \boldsymbol{w},\, \boldsymbol{\gamma}_T^\times \boldsymbol{w} \ \in\ \mathbf{H}_{\boldsymbol{\tau}}^{-1/2}(\partial\Omega) := \{\boldsymbol{\psi} \in \mathbf{H}^{-1/2}(\partial\Omega) : \boldsymbol{\psi} \cdot \boldsymbol{n} = 0 \text{ a.e. on } \partial\Omega\}$$

However, $\boldsymbol{\gamma}_T$, $\boldsymbol{\gamma}_T^\times$ still not surjective w.r.t. this space.

Definition 2.27. $\mathbf{Y}_T^\times := \boldsymbol{\gamma}_T^\times(\mathbf{H}(\mathbf{curl}, \Omega))$, $\|\boldsymbol{s}\|_{\mathbf{Y}_T^\times} := \inf\limits_{\substack{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}, \Omega) \\ \boldsymbol{\gamma}_T^\times \boldsymbol{u} = \boldsymbol{s}}} \|\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl}, \Omega)}$

Theorem 2.28.

(i) $\mathbf{Y}_T^\times$ is a Hilbert space

(ii) $\boldsymbol{\gamma}_T^\times : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{Y}_T^\times$ is surjective and has a bounded right inverse

(iii) The map $\boldsymbol{\gamma}_T : \mathbf{H}(\mathbf{curl}, \Omega) \to (\mathbf{Y}_T^\times)^*$ is well-defined

(iv) For any $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbf{H}(\mathbf{curl}, \Omega)$,

$$\int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \cdot \boldsymbol{v}\, dx \ = \ \int_\Omega \boldsymbol{w} \cdot \mathbf{curl}\, \boldsymbol{v}\, dx - \langle \boldsymbol{\gamma}_T^\times \boldsymbol{w},\, \boldsymbol{\gamma}_T \boldsymbol{v} \rangle_{\mathbf{Y}_T^\times \times (\mathbf{Y}_T^\times)^*}$$

Remark. One can also show that $\boldsymbol{\gamma}_T : \mathbf{H}(\mathbf{curl}, \Omega) \to (\mathbf{Y}_T^\times)^*$ is surjective, and so $\mathbf{Y}_T$ and $\mathbf{Y}_T^\times$ are dual to each other.

Lemma 2.29.

$$\begin{aligned}\mathbf{H}_0(\mathbf{curl}, \Omega) \ := \ \overline{\mathbf{C}_c^\infty(\Omega)}^{\|\cdot\|_{\mathbf{H}(\mathbf{curl}, \Omega)}} \ &= \ \{\boldsymbol{w} \in \mathbf{H}(\mathbf{curl}, \Omega) : \boldsymbol{\gamma}_T \boldsymbol{w} = 0\} \\ &= \ \{\boldsymbol{w} \in \mathbf{H}(\mathbf{curl}, \Omega) : \boldsymbol{\gamma}_T^\times \boldsymbol{w} = 0\}\end{aligned}$$

[1] Short hands: $\mathbf{C}^k(D) := C^k(D)^3$, $\mathbf{L}^2(D) := L^2(D)^3$, $\mathbf{H}^s(D) := H^s(D)^3$

The following is additional information (not part of the lecture)

Proof of Theorem 2.28.
Part (i):

$$\boldsymbol{s} \in \mathbf{Y}_T^\times, \qquad \boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega), \boldsymbol{\gamma}_T^\times \boldsymbol{v} = \boldsymbol{s}$$

For $\boldsymbol{\phi} \in \mathbf{H}(\mathbf{curl}, \Omega)$, define linear functional

$$L_{\boldsymbol{s}}(\boldsymbol{\phi}) := \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{\phi}\, dx - \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\phi}\, dx$$

Properties: (exercise!)

- $L_{\boldsymbol{s}}$ depends only on $\boldsymbol{s}$
- $|L_{\boldsymbol{s}}| \leq \|\boldsymbol{s}\|_{\mathbf{Y}_T^\times} \|\boldsymbol{\phi}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$

This shows that $L_{\boldsymbol{s}} \in \mathbf{H}(\mathbf{curl}, \Omega)^*$.
Due to Riesz' theorem, there exists $\boldsymbol{w}_{\boldsymbol{s}} \in \mathbf{H}(\mathbf{curl}, \Omega)$:

$$L_{\boldsymbol{s}}(\boldsymbol{\phi}) = (\boldsymbol{w}_{\boldsymbol{s}}, \boldsymbol{\phi})_{\mathbf{H}(\mathbf{curl},\Omega)} = (\mathbf{curl}\, \boldsymbol{w}_{\boldsymbol{s}}, \mathbf{curl}\, \boldsymbol{\phi})_{\boldsymbol{L}^2(\Omega)} + (\boldsymbol{w}_{\boldsymbol{s}}, \boldsymbol{\phi})_{\boldsymbol{L}^2(\Omega)}$$

and $\|\boldsymbol{w}_{\boldsymbol{s}}\|_{\mathbf{H}(\mathbf{curl},()\Omega)} = \|L_{\boldsymbol{s}}\|_{\mathbf{H}(\mathbf{curl},\Omega)^*}$
Further properties: (exercise!)

- $\mathbf{curl}\, \boldsymbol{w}_{\boldsymbol{s}} = -\boldsymbol{w}_{\boldsymbol{s}} \in \mathbf{H}(\mathbf{curl}, \Omega)$ *Hint:* test with $\boldsymbol{\phi} \in \mathbf{C}_c^\infty(\Omega)$
- $\boldsymbol{\gamma}_T^\times(\mathbf{curl}\, \boldsymbol{w}_{\boldsymbol{s}}) = \boldsymbol{s}$ *Hint:* $\boldsymbol{\phi} \in \mathbf{H}^1(\Omega)$, use Thm.2.26

Then

$$\|\boldsymbol{s}\|_{\mathbf{Y}_T^\times} \leq \|\mathbf{curl}\, \boldsymbol{w}_{\boldsymbol{s}}\|_{\mathbf{H}(\mathbf{curl},\Omega)} = \|\boldsymbol{w}_{\boldsymbol{s}}\|_{\mathbf{H}(\mathbf{curl},\Omega)} = \|L_{\boldsymbol{s}}\|_{\mathbf{H}(\mathbf{curl},\Omega)^*} \leq \|\boldsymbol{s}\|_{\mathbf{Y}_T^\times}$$

$\Longrightarrow$ Isometry between $\mathbf{Y}_T^\times$ and closed (!) subspace of $\mathbf{H}(\mathbf{curl}, \Omega)$ □

Part (ii) follows by construction □

Part (iii) For smooth $\boldsymbol{s}$, $\boldsymbol{\phi}$:

$$\langle \boldsymbol{\gamma}_T \boldsymbol{\phi}, \boldsymbol{s} \rangle_{(\mathbf{Y}_T^\times)^* \times \mathbf{Y}_T^\times} \overset{\text{Thm.2.26}}{=} L_{\boldsymbol{s}}(\boldsymbol{\phi})$$

Properties of $L_{\boldsymbol{s}}$ + density $\Longrightarrow$ $\boldsymbol{\gamma}_T \boldsymbol{\phi} \in (\mathbf{Y}_T^\times)^*$ bounded in $\|\boldsymbol{\phi}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$ □
Part (iv) follows from the above □

Characterization of the two trace spaces

Observation: $\boldsymbol{\gamma}_T^\times \boldsymbol{w}$ could have additional smoothness: for $u \in C^\infty(\overline{\Omega})$,

$$\langle \boldsymbol{\gamma}_T^\times \boldsymbol{w}, \underbrace{\boldsymbol{\gamma}_T(\nabla u)}_{=??} \rangle_{\partial\Omega} \overset{\text{L.2.1 + density}}{=} \int_\Omega \boldsymbol{w} \cdot \underbrace{\mathbf{curl}\, \nabla u}_{=0}\, dx - \int_\Omega (\underbrace{\mathbf{curl}\, \boldsymbol{w}}_{\in \mathbf{H}(\mathrm{div}\,, \Omega)}) \cdot \nabla u \, dx$$

$$\overset{\text{L.2.24}}{=} \int_\Omega (\underbrace{\mathrm{div}\, \mathbf{curl}\, \boldsymbol{w}}_{=0})\, u \, dx - \langle \underbrace{\gamma_n(\mathbf{curl}\, \boldsymbol{w})}_{=??}, \gamma u \rangle_{\partial\Omega}$$

Differential operators on the surface (part I)

- **Surface gradient** $\nabla_{\partial\Omega}$: for $u \in C^\infty(\overline{\Omega})$, $\varphi = \gamma u$,

  $$\nabla_{\partial\Omega} \varphi := \boldsymbol{\gamma}_T(\nabla u) = \boldsymbol{\gamma} \nabla u - \frac{\partial u}{\partial \boldsymbol{n}} \boldsymbol{n}$$

  Definition only depends on φ. Extension: $\nabla_{\partial\Omega} : H^{3/2}(\partial\Omega) \to \mathbf{H}_{\boldsymbol{\tau}}^{1/2}(\partial\Omega)$
  For $\Omega = \mathbb{R}^2 \times \mathbb{R}^-$, $\nabla_{\partial\Omega} \varphi = (\partial_1 \varphi, \partial_2 \varphi, 0)^T$

- **Surface divergence** $\mathrm{div}_{\partial\Omega} : \mathbf{H}_{\boldsymbol{\tau}}^{-1/2}(\partial\Omega) \to H^{-3/2}(\partial\Omega)$
  defined by duality:

  $$\langle \mathrm{div}_{\partial\Omega} \boldsymbol{v}, \varphi \rangle_{\partial\Omega} = -\langle \boldsymbol{v}, \nabla_{\partial\Omega} \varphi \rangle_{\partial\Omega}$$

  For $\Omega = \mathbb{R}^2 \times \mathbb{R}^-$, $\mathrm{div}_{\partial\Omega} \boldsymbol{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$

We have shown above that

$$\langle \mathrm{div}_{\partial\Omega} \boldsymbol{\gamma}_T^\times \boldsymbol{w}, \gamma u \rangle_{\partial\Omega} = \langle \boldsymbol{\gamma}_T^\times \boldsymbol{w}, \underbrace{\nabla_{\partial\Omega} \gamma u}_{=\boldsymbol{\gamma}_T(\nabla u)} \rangle_{\partial\Omega} = \langle \underbrace{\gamma_n(\mathbf{curl}\, \boldsymbol{w})}_{\in \mathbf{H}^{-1/2}(\partial\Omega)}, \gamma u \rangle_{\partial\Omega}$$

Hence: $\quad \mathrm{div}_{\partial\Omega} \boldsymbol{\gamma}_T^\times \boldsymbol{w} \in \mathbf{H}^{-1/2}(\partial\Omega) \quad !$

Lemma (without proof):
For C^1-domains Ω,

$$\mathbf{Y}_T^\times = \mathbf{H}^{-1/2}(\mathrm{Div}, \partial\Omega) := \{ \boldsymbol{\psi} \in \mathbf{H}_{\boldsymbol{\tau}}^{-1/2}(\partial\Omega) : \mathrm{div}_{\partial\Omega} \boldsymbol{\psi} \in H^{-1/2}(\partial\Omega) \}$$

Differential operators on the surface (part II)

- **Vectorial surface curl:**

$$\mathbf{curl}_{\partial\Omega}\varphi \; := \; -\boldsymbol{n} \times \nabla_{\partial\Omega}\varphi$$

  For $\Omega = \mathbb{R}^2 \times \mathbb{R}^-$, $\mathbf{curl}_{\partial\Omega}\varphi \; = \; (-\partial_2\varphi,\, \partial_1\varphi,\, 0)^T$

- **Scalar surface curl**:

$$\mathrm{curl}_{\partial\Omega}\boldsymbol{v} \; := \; -\mathrm{div}_{\partial\Omega}(\boldsymbol{n} \times \boldsymbol{v})$$

  For $\Omega = \mathbb{R}^2 \times \mathbb{R}^-$, $\mathrm{curl}_{\partial\Omega}\boldsymbol{v} \; = \; \partial_2 v_1 - \partial_1 v_2$

Lemma (without proof):
For C^1-domains Ω,

$$\mathbf{Y}_T \; = \; \mathbf{H}^{-1/2}(\mathrm{Curl}, \partial\Omega) \; := \; \{\boldsymbol{\psi} \in \mathbf{H}_{\boldsymbol{\tau}}^{-1/2}(\partial\Omega) : \mathrm{curl}_{\partial\Omega}\boldsymbol{\psi} \in H^{-1/2}(\partial\Omega)\},$$

is surjective and has a bounded right inverse.

Further characterizations:

- Lipschitz polyhedra: continuity properties needed across edges $\rightsquigarrow$

$$\mathbf{H}_{\perp}^{-1/2}(\mathrm{Curl}, \partial\Omega), \qquad \mathbf{H}_{\|}^{-1/2}(\mathrm{Div}, \partial\Omega)$$

  [Buffa & Ciarlet, M2AN 24:9–30, 2001].

- General Lipschitz domains:
  [Buffa, Costabel, & Sheen, JMAA 276:845–867, 2002].