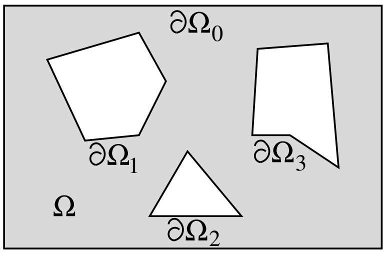
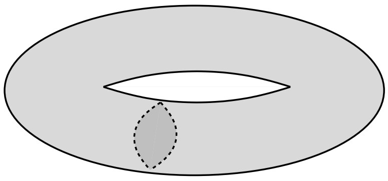
Helmholtz decompositions

Assumptions on the domain. Let Ω be a Lipschitz domain such that $\partial\Omega = \bigcup_{j=0}^{J} \partial\Omega_j$, where $\partial\Omega_j$ are the connected components, and $\partial\Omega_0$ is the boundary of the unbounded component of $\mathbb{R}^3 \setminus \overline{\Omega}$ (J is the number of holes).





Furthermore, let (the *first Betti number*) L be the minimal number such that there exists L *interior cuts* Σ_ℓ, $\ell = 1, \ldots, L$ such that

- $\{\Sigma_\ell\}_{\ell=1}^{L}$ pairwise disjoint
- each Σ_ℓ is open connected part of a smooth surface and $\partial\Sigma_\ell \subset \partial\Omega$,
- $\Omega \setminus \bigcup_{\ell=1}^{L} \Sigma_\ell$ is simply-connected, pseudo Lipschitz domain.

Examples: torus: $J = 0$, $L = 1$; simply-connected domains: $L = 0$, convex domains: $J = 0$, $L = 0$

Theorem (see Monk, Thm.3.41–3.44)

(i) $$\ker(\mathbf{curl}_{|\mathbf{H}_0(\mathbf{curl},\Omega)}) = \nabla H_0^1(\Omega) \oplus \mathbf{K}_N(\Omega)$$
with the *normal cohomology space*

$$\mathbf{K}_N(\Omega) := \{\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega) : \mathbf{curl}\, \boldsymbol{v} = 0,\ \operatorname{div} \boldsymbol{v} = 0\},$$
$$\dim(\mathbf{K}_N(\Omega)) = J.$$

(ii) $$\ker(\operatorname{div}_{|\mathbf{H}_0(\operatorname{div},\Omega)}) = \mathbf{curl}\, \mathbf{H}_0(\mathbf{curl}, \Omega) \oplus \mathbf{K}_T(\Omega)$$
with the *tangential cohomology space*

$$\mathbf{K}_T(\Omega) := \{\boldsymbol{v} \in \mathbf{H}_0(\operatorname{div}, \Omega) : \operatorname{div} \boldsymbol{v} = 0,\ \mathbf{curl}\, \boldsymbol{v} = 0\},$$
$$\dim(\mathbf{K}_T(\Omega)) = L.$$

Theorem (see Monk, Thm.3.45)
For any $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$ there exists the unique decomposition

$$\boldsymbol{u} = \nabla\varphi + \mathbf{curl}\, \boldsymbol{w} + \boldsymbol{f}_N\,,$$

where $\varphi \in H_0^1(\Omega)$, $\boldsymbol{f}_N \in \mathbf{K}_N(\Omega)$ and

$$\boldsymbol{w} \in \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}_0(\operatorname{div}, \Omega) \text{ with } \langle \gamma_n \boldsymbol{w},\, 1\rangle_{\Sigma_\ell} = 0 \quad \forall \ell = 1, \ldots, L.$$

A similar decomposition holds with $\varphi \in H^1(\Omega)$, $\boldsymbol{w} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$, and $\boldsymbol{f}_T \in \mathbf{K}_T(\Omega)$.