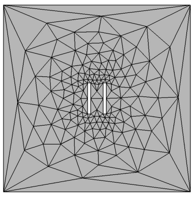
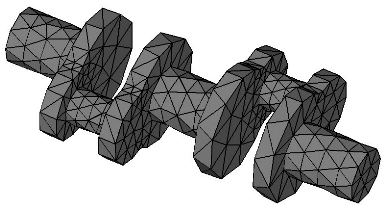
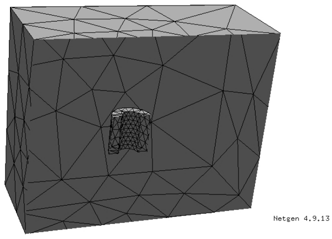
Triangulations

$\Omega \subset \mathbb{R}^d$ bounded Lipschitz polytype ($d = 2$: polygon, $d = 3$: polyhedron)

$\mathcal{T}^h(\Omega)$ *regular triangulation*, i.e.,

- $\mathcal{T}^h(\Omega) = \{T\}$, where T non-degenerate simplices ($d = 2$: triangles, $d = 3$: tetrahedra), $T = \overline{T}$
- $\overline{\Omega} = \bigcup_{T \in \mathcal{T}^h(\Omega)} T$
- the intersection of two different elements is either empty, one edge or one face of both elements

Examples:

We define

$$h_T := \operatorname{diam}(T), \qquad h := \max_{T \in \mathcal{T}^h(\Omega)} h_T$$

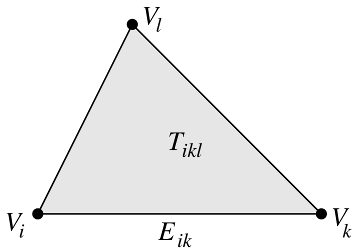
$$\rho_T := \text{radius of larges ball inscribed in } T$$

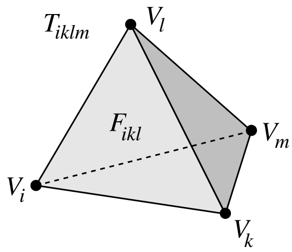
A family of triangulations $\{\mathcal{T}^h(\Omega)\}_{h \in \Theta}$ is called

- *shape regular* iff $\exists c > 0 \ \forall h \in \Theta \ \forall T \in \mathcal{T}^h(\Omega) : \rho_T \geq c\, h_T$
- *quasi uniform* iff shape reg. and $\exists c > 0 \ \forall h \in \Theta \ \forall T \in \mathcal{T}^h(\Omega) : h_T \geq c\, h$

For a (fixed) triangulation we define

set of *vertices* :	$\mathcal{V} = \{V_i\}$	
set of *edges* :	$\mathcal{E} = \{E_{ik}\}$	
set of *triangles* :	$\mathcal{T} = \{T_{ik\ell}\}$	(for $d = 2$)
set of *faces* :	$\mathcal{F} = \{F_{ik\ell}\}$	(for $d = 3$)
set of *tetrahedra* :	$\mathcal{T} = \{T_{ik\ell m}\}$	(for $d = 3$)





Finite Elements

Definition 3.1. A *finite element* $(T, V, \mathcal{N})$ consists of

- geometric domain T (*element domain*)
- finite-dimensional space V of functions on T (*space of shape functions*)
- set $\mathcal{N}$ of linearly independent functionals (*set of nodal variables*) which form a basis of V^*

Lemma 3.2. Let T, V be as in Def. 3.1 and $\mathcal{N} = \{\psi_1, \ldots, \psi_N\} \subset V^*$. Then the following statements are equivalent:

- $\mathcal{N}$ is a basis of V^*
- $\forall v \in V : [\forall i = 1, \ldots, N : \psi_i(v) = 0] \implies v = 0$ ($\mathcal{N}$ determines V)
- $\forall \beta_1, \ldots, \beta_N \in \mathbb{R} \quad \exists!\ v \in V : \psi_i(v) = \beta_i \quad \forall i = 1, \ldots, N$ (unisolvence)

Definition 3.3. Let $(T, V, \mathcal{N})$ be a finite element with $\mathcal{N} = \{\psi_1, \ldots, \psi_N\}$. The (unique) basis $\{\varphi_1, \ldots, \varphi_N\}$ of V fulfilling

$$\psi_i(\varphi_k) \ = \ \delta_{ik}$$

is called *nodal basis.*

Definition 3.4. Two finite elements $(\widehat{T}, \widehat{V}, \widehat{\mathcal{N}})$ and $(T, V, \mathcal{N})$ are called *affine equivalent* if there exists an affine linear map $\Phi(x) = F\,x + b$ with

- $T = \Phi(\widehat{T})$
- $V \circ \Phi = \widehat{V}$ ($\widehat{V}$ is pull-back of V)
- $\mathcal{N} = \{v \mapsto \widehat{\psi}(v \circ \Phi) : \widehat{\psi} \in \widehat{\mathcal{N}}\}$ ($\mathcal{N}$ is push-forward of $\widehat{\mathcal{N}}$)

References:
P. G. Ciarlet, The Finite Element Method for Elliptic Problems, SIAM
S. C. Brenner and L. R. Scott, The Mathematical Theory of Finite Element Methods, Springer