Existence and Uniqueness Theorems

Theorem A (Lax-Milgram).
Let V be a Hilbert space and let $a : V \times V \to \mathbb{R}$ be a bilinear form that is

- *bounded* on V, i.e. $\exists C_a < \infty$: $a(u, v) \le C_a \|u\|_V \|v\|_V$ for all $u, v \in V$.
- *coercive* on V, i.e. $\exists c_a > 0 : a(v, v) \ge c_a \|v\|_V^2$ for all $v \in V$,

Then for any $\ell \in V^*$, there exists a unique solution $u \in V$:

$$a(u, v) \;=\; \langle \ell, v\rangle \qquad \forall v \in V,$$

and

$$\|u\|_V \;\le\; \frac{1}{c_a}\,\|\ell\|_{V^*}$$

Proof: Lecture on Numerical Methods for PDEs or [Brenner & Scott: The Mathematical Theory of Finite Element Methods, Springer-Verlag]

Theorem B (Babuška-Aziz).
Let U and V be Hilbert spaces and let $b : U \times V \to \mathbb{R}$ be a bilinear form such that

- b is *bounded*, i.e. $\exists C_b < \infty : b(u, v) \le C_b \|u\|_U \|v\|_V$ for all $u \in U$, $v \in V$,
- $\exists \beta > 0$: $\inf_{u \in U} \sup_{v \in V} \frac{b(u, v)}{\|u\|_U \|v\|_V} \ge \beta$,
- for each $v \in V \setminus \{0\}$ there exists $u \in U$ with $b(u, v) \ne 0$.

Then for any $\ell \in V^*$, there exists a unique solution $u \in U$:

$$b(u, v) \;=\; \langle \ell, v\rangle \qquad \forall v \in V,$$

and

$$\|u\|_U \;\le\; \frac{1}{\beta}\,\|\ell\|_{V^*}$$

Proof: Lecture on Numerical Methods for Continuum Mechanics or [Girault & Raviart, Finite Element Methods for Navier-Stokes Equations, Springer-Verlag]

Theorem C (Brezzi).
Let V, Q be Hilbert spaces and let $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ be bilinear forms such that

- a is *bounded*, i.e. $\exists C_a < \infty$: $a(v, w) \leq \|v\|_V \|w\|_V$ for all $v, w \in V$,
- b is *bounded*, i.e. $\exists C_b < \infty$: $b(v, q) \leq \|v\|_V \|q\|_Q$ for all $v \in V,\ q \in Q$,
- a is *coercive on* $\ker B := \{v \in V : b(v, q) = 0\ \forall q \in Q\}$,
  i.e. $\exists \alpha > 0$: $a(v, v) \geq \alpha \|v\|_V^2$ for all $v \in \ker B$,
- $\exists \beta > 0$: $\inf\limits_{q \in Q} \sup\limits_{v \in V} \dfrac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta$ (Ladyshenskaya-Babuška-Brezzi -condition).

Then, for any $f \in V^*$ and $g \in Q^*$, there exists a unique solution $(u, p) \in V \times Q$:

$$
\begin{aligned}
a(u, v) + b(v, p) &= \langle f, v \rangle \qquad \forall v \in V, \\
b(u, q) &= \langle g, q \rangle \qquad \forall q \in Q,
\end{aligned}
$$

and

$$
\begin{aligned}
\|u\|_V &\leq \frac{1}{\alpha} \|f\|_{V^*} + \frac{1}{\beta} \Big(1 + \frac{C_a}{\alpha}\Big) \|g\|_{Q^*}\,, \\
\|q\|_Q &\leq \frac{1}{\beta} \Big(1 + \frac{C_a}{\alpha}\Big) \|f\|_{V^*} + \frac{C_a}{\beta^2} \Big(1 + \frac{C_a}{\alpha}\Big) \|g\|_{Q^*}\,.
\end{aligned}
$$

Proof: Lecture on Numerical Methods for PDEs or [Brenner & Scott] or [Girault & Raviart] or [Brezzi & Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag]

Theorem D (Fredholm).
Let X be a Hilbert space and $B : X \to X$ a bounded linear operator. Suppose that $B = \mathrm{id} + A$, where A is a compact operator. Then either

- The homogeneous equation $B\,u = 0$ has only the trivial solution $u = 0$ in X. In this case, for every $f \in X$, the inhomogeneous equation $B\,u = f$ has a unique solution depending continuously on f; or
- The homogeneous equation $B\,u = 0$ has exactly m linearly independent solutions for some finite integer $m > 0$.

Proof: e.g. [McLean, Strongly Elliptic Systems and Boundary Integral Equations]