Abstract Error Estimates

Lemma A* (Céa).
Let $V_h \subset V$ be Hilbert spaces and let $a : V \times V \to \mathbb{R}$ be a bilinear form that is

- *bounded* on V, i.e. $\exists C_a < \infty$: $a(u, v) \leq C_a \|u\|_V \|v\|_V$ for all $u, v \in V$.
- *coercive* on V, i.e. $\exists c_a > 0 : a(v, v) \geq c_a \|v\|_V^2$ for all $v \in V$.

For fixed $\ell \in V^*$, let $u \in V$ and $u_h \in V_h$ be the (unique) solutions to

$$\begin{aligned} a(u, v) &= \langle \ell, v \rangle \qquad \forall v \in V, \\ a(u_h, v_h) &= \langle \ell, v_h \rangle \qquad \forall v_h \in V_h\,. \end{aligned}$$

Then

$$\|u - u_h\|_V \;\leq\; \frac{C_a}{c_a} \inf_{w_h \in V_h} \|u - w_h\|_V\,.$$

Proof: Galerkin orthogonality: for all $v_h \in V_h : a(u - u_h, v_h) = 0$. Hence, for any $w_h \in V_h$:

$$\|u-u_h\|_V^2 \;\leq\; c_a^{-1}\, a(u-u_h, u-u_h) \;=\; c_a^{-1}\, a(u-u_h, u-w_h) \;\leq\; \frac{C_a}{c_a} \|u-u_h\|_V \|u-w_h\|_V \qquad \square$$

Lemma B*.
Let $U_h \subset U$ and $V_h \subset V$ be Hilbert spaces and let $b : U \times V \to \mathbb{R}$ be a bilinear form such that

- $\exists C_b < \infty : b(u, v) \leq C_b \|u\|_U \|v\|_V$ for all $u \in U,\ v \in V$, *(b is bounded)*
- $\exists \beta > 0$: $\displaystyle\inf_{u \in U} \sup_{v \in V} \frac{b(u, v)}{\|u\|_U \|v\|_V} \geq \beta$ *(continuous inf-sup condition)*
  $\exists \beta_h > 0$: $\displaystyle\inf_{u_h \in U_h} \sup_{v_h \in V_h} \frac{b(u_h, v_h)}{\|u_h\|_U \|v_h\|_V} \geq \beta_h$ *(discrete inf-sup condition)*
- for each $v \in V \setminus \{0\}$ there exists $u \in U$ with $b(u, v) \neq 0$,
  for each $v_h \in V_h \setminus \{0\}$ there exists $u_h \in U_h$ with $b(u_h, v_h) \neq 0$.

For fixed $\ell \in V^*$, let $u \in U$, $u_h \in U_h$ be the (unique) solutions to

$$\begin{aligned} b(u, v) &= \langle \ell, v \rangle \qquad \forall v \in V, \\ b(u_h, v_h) &= \langle \ell, v_h \rangle \qquad \forall v_h \in V_h\,. \end{aligned}$$

Then

$$\|u - u_h\|_U \;\leq\; \left(1 + \frac{C_b}{\beta_h}\right) \inf_{w_h \in U_h} \|u - w_h\|_U\,.$$

Proof: [see lecture]

Lemma C*.
Let $V_h \subset V$, $Q_h \subset Q$ be Hilbert spaces and let $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ be bilinear forms such that

- $\exists C_a < \infty$: $a(v,\, w) \le \|v\|_V \|w\|_V$ for all $v,\, w \in V$, *(a is bounded)*
- $\exists C_b < \infty$: $b(v,\, q) \le \|v\|_V \|q\|_Q$ for all $v \in V,\ q \in Q$, *(b is bounded)*
- $\exists \alpha > 0 : \quad a(v,\, v) \ge \alpha \, \|v\|_V^2$ *(continuous kernel-coerciveness)*
  $\forall v \in \ker B := \{v \in V : b(v,\, q) = 0 \, \forall q \in Q\}$,
- $\exists \alpha_h > 0 : \quad a(v_h,\, v_h) \ge \alpha_h \, \|v_h\|_V^2$ *(discrete kernel-coerciveness)*
  $\forall v_h \in \ker B_h := \{v_h \in V_h : b(v_h,\, q_h) = 0 \, \forall q_h \in Q_h\}$,
- $\exists \beta > 0$: $\inf\limits_{q \in Q} \sup\limits_{v \in V} \dfrac{b(v,\, q)}{\|v\|_V \|q\|_Q} \ge \beta$ *(continuous inf-sup condition)*
- $\exists \beta_h > 0$: $\inf\limits_{q_h \in Q_h} \sup\limits_{v_h \in V_h} \dfrac{b(v_h,\, q_h)}{\|v_h\|_V \|q_h\|_Q} \ge \beta_h$ *(discrete inf-sup condition)*

For fixed $f \in V^*$, $g \in Q^*$, let $(u,\, v) \in V \times Q$ and $(u_h,\, v_h) \in V_h \times Q_h$ be the solutions to

$$\begin{aligned} a(u,\, v) + b(v,\, p) &= \langle f,\, v\rangle && \forall v \in V, \\ b(u,\, q) &= \langle g,\, q\rangle && \forall q \in Q, \\ \\ a(u_h,\, v_h) + b(v_h,\, p_h) &= \langle f,\, v_h\rangle && \forall v_h \in V_h\,, \\ b(u_h,\, q_h) &= \langle g,\, q_h\rangle && \forall q_h \in Q_h\,. \end{aligned}$$

Then

$$\|u - u_h\|_V \le \big(1 + \tfrac{C_a}{\alpha_h}\big)\big(1 + \tfrac{C_b}{\beta_h}\big) \inf_{w_h \in V_h} \|u - w_h\|_V + \tfrac{C_b}{\alpha_h} \inf_{r_h \in Q_h} \|p_h - r_h\|_Q$$

$$\|p - q_h\|_Q \le \big(1 + \tfrac{C_a}{\alpha_h}\big)\big(1 + \tfrac{C_b}{\beta_h}\big) \inf_{w_h \in V_h} \|u - w_h\|_V + \big(1 + \tfrac{C_b}{\beta_h}\big(1 + \tfrac{C_a}{\alpha_h}\big)\big) \inf_{r_h \in Q_h} \|p_h - r_h\|_Q$$

Proof: Lecture on Numerical Methods for Continuum Mechanics or [Brenner & Scott] or [Girault & Raviart] or [Brezzi & Fortin]