Additive Schwarz Preconditioners

Problem setting:

- finite-dimensional Hilbert space V,
- coercive bilinear form $a(\cdot, \cdot) : V \times V \to \mathbb{R}$
- associated operator $A : V \to V^*$, defined by $\langle A\, v,\, w\rangle = a(v,\, w)$
- problem to solve: $\quad A\, u \;=\; f$

Additive Schwarz Preconditioner:

- subspaces: $V_i \subset V$:

$$V \;=\; \sum_{i=1}^{n} V_i$$

- prolongation (embedding) operators $R_i^\top$ and corresponding adjoints:

$$R_i^\top : V_i \to V, \qquad\qquad R_i : V^* \to V_i^*$$

- local operators ("exact solvers"):

$$A_i := R_i A\, R_i^\top : V_i \to V_i^*$$

- Additive Schwarz preconditioner:

$$C_{\mathrm{AS}}^{-1} \;:=\; \sum_{i=1}^{n} R_i^\top\, A_i^{-1} R_i$$

Additive Schwarz Lemma. C_{AS}^{-1} is the inverse of a self-adjoint and positive definite operator $C_{\mathrm{AS}} : V \to V^*$, and

$$\langle C_{\mathrm{AS}}\, u,\, u\rangle \;=\; \min_{\substack{u_i \in V_i \\ u = \sum_{i=1}^n u_i}} \sum_{i=1}^{n} \|u_i\|_a^2 \qquad \text{where } \|v\|_a := \sqrt{a(v,\, v)}$$

Corollary. If there exist constants γ_1, γ_2 with

$$\gamma_1\, \|u\|_a^2 \;\le\; \min_{\substack{u_i \in V_i \\ u = \sum_{i=1}^n u_i}} \sum_{i=1}^{n} \|u_i\|_a^2 \;\le\; \gamma_2\, \|u\|_a^2\,,$$

then

$$\mathrm{cond}(C_{\mathrm{AS}}^{-1} A) \;\le\; \frac{\gamma_2}{\gamma_1}\,.$$

Remark: γ_2 is called constant of stable splitting. A bound for γ_1 can often be obtained by a coloring argument. See, e.g., [Brenner & Scott] or [Toselli & Widlund, Domain Decomposition Methods – Algorithms and Theory, Springer, 2005]