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19 Consider the following variational problems:

Find $\boldsymbol{u} \in H(\mathbf{curl}\,, \Omega)$ such that

$$\int_\Omega \nu \, \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx \quad \forall \boldsymbol{v} \in H(\mathbf{curl}\,, \Omega). \tag{7.1}$$

Find $(\boldsymbol{u}, \varphi) \in H(\mathbf{curl}\,, \Omega) \times H^1(\Omega)|\mathbb{R}$ such that

$$\begin{aligned} \int_\Omega \nu \, \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} \, dx + \int_\Omega \boldsymbol{v} \cdot \nabla \varphi \, dx &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx \\ \int_\Omega \boldsymbol{u} \cdot \nabla \psi \, dx &= 0 \end{aligned} \tag{7.2}$$

for all $(\boldsymbol{v}, \psi) \in H(\mathbf{curl}\,, \Omega) \times H^1(\Omega)|\mathbb{R}$.

Show that if $\boldsymbol{f} \perp \nabla H^1(\Omega)$, then for any solution $(\boldsymbol{u}, \varphi)$ of (7.2) we have that

- $\nabla \varphi = 0$,
- $\boldsymbol{u}$ solves (7.1).

20 Show Lemma 3.2 from the lecture, i.e. that the statements

- $\mathcal{N}$ is a basis of V^*
- $\forall v \in V : [\forall i = 1, ..., N : \psi_i(v) = 0] \Longrightarrow v = 0$
- $\forall \beta_1, ..., \beta_N \in \mathbb{R} \quad \exists! v \in V : \psi_i(v) = \beta_i \quad \forall i = 1, ..., N$

are equivalent.

Hint: Relate each statement to a statement for the matrix $G \in \mathbb{R}^{N \times N}$,

$$G_{ij} = \psi_i(\phi_j),$$

where $\{\phi_j\}_{j=1}^N$ is a fixed basis of V.

21 Show that the triangular Nédélec element (Definition 3.8) is indeed a finite element (according to Definition 3.1).

Hint: Use the second line in Lemma 3.2 (Exercise 20) and Lemma 3.10 (iii).