16 Show that

(a) $\forall\, \phi \in H_0^1(\Omega) : \boldsymbol{\gamma}_T(\nabla\phi) = 0$,

(b) $\forall\, \boldsymbol{w} \in H_0(\mathbf{curl}\,, \Omega) : \gamma_n(\mathbf{curl}\,\boldsymbol{w}) = 0$,

i.e.

$$H_0^1(\Omega) \xrightarrow{\nabla} H_0(\mathbf{curl}\,, \Omega) \xrightarrow{\mathbf{curl}} H_0(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega).$$

Hint: Show the identities for $\phi \in C_0^\infty(\Omega)$ and $\boldsymbol{w} \in C_0^\infty(\Omega)^3$, use integration by parts and use density.

17 Prove Lemma 2.33 (iii) from the lecture, i.e. show that for each $\boldsymbol{q} \in H(\mathrm{div}, \Omega)$ with $\mathrm{div}\ \boldsymbol{q} = 0$, $\gamma_n\, \boldsymbol{q} = 0$, there exists $\boldsymbol{w} \in H_0(\mathbf{curl}\,, \Omega)$ with

$$\boldsymbol{q} = \mathbf{curl}\,\boldsymbol{w}, \quad \mathrm{div}\ \boldsymbol{w} = 0, \quad \|\boldsymbol{w}\|_{H(\mathbf{curl},\Omega)} \le C\|\boldsymbol{q}\|_{L^2(\Omega)^3}.$$

Hint: Set $\boldsymbol{w}_2 \in H^1(\mathbb{R}^3)$ as in the proof of Lemma 2.33 (ii) and let $\psi \in H_0^1(\Omega)$ be such that

$$\int_\Omega \nabla\psi \cdot \nabla v\, dx = \int_\Omega \boldsymbol{w}_2 \cdot \nabla v\, dx \quad \forall v \in H_0^1(\Omega).$$

Set $\boldsymbol{w} := \boldsymbol{w}_2 - \nabla\psi$ and show the properties. Show and use that

$$\|\nabla\psi\|_{L^2(\Omega)^3} \le \|\boldsymbol{w}_2\|_{L^2(\Omega)^3}.$$

18 Let $\boldsymbol{q} \in H(\mathrm{div}, \Omega)$ with $\mathrm{div}\ \boldsymbol{q} = 0$ and let $\tilde{\Omega} \supset \Omega$.
Show that $\exists\, \tilde{\boldsymbol{q}} \in H(\mathrm{div}, \tilde{\Omega})$:

$$\begin{aligned} \tilde{\boldsymbol{q}} &= \boldsymbol{q} \quad && \text{in } \Omega, \\ \mathrm{div}\ \tilde{\boldsymbol{q}} &= 0, && \\ \gamma_n\, \tilde{\boldsymbol{q}} &= 0 && \text{on } \partial\tilde{\Omega}. \end{aligned}$$

Hint: How should you choose $\gamma_n\, \tilde{\boldsymbol{q}}$ on $\partial\Omega$? Use Exercise 13 on the domain $\tilde{\Omega}\backslash\bar{\Omega}$.