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**13** Prove the **Remark** after Theorem 2.22 from the lecture, i.e.

If $\psi \in H^{-1/2}(\partial\Omega)$ with $\langle \psi, 1\rangle_{\partial\Omega} = 0$, then there exists a $\boldsymbol{q} \in H(\mathrm{div}, \Omega)$ with

$$\begin{aligned} &\mathrm{div}\, \boldsymbol{q} = 0, \\ &\gamma_n\, \boldsymbol{q} = \psi, \\ &\|\boldsymbol{q}\|_{H(\mathrm{div},\Omega)} \leq C\|\psi\|_{H^{-1/2}(\partial\Omega)}. \end{aligned}$$

*Hint:* Prove the remark analogously as the proof of Theorem 2.22 but solve

$$-\triangle u = 0, \qquad \frac{\partial u}{\partial n} = \psi.$$

**14** Let us define

$$\begin{aligned} V &:= \overline{C_0^\infty(\Omega)^3}^{\|\cdot\|_{H(\mathbf{curl})}}, \\ W &:= \{\boldsymbol{w} \in H(\mathbf{curl}\,, \Omega) : \boldsymbol{\gamma}_T^\times\, \boldsymbol{w} = 0\}. \end{aligned}$$

Show that $V = W$.

*Hint:* Remember that $H(\mathbf{curl}\,, \Omega) = V \oplus V^\perp$ (where $V^\perp$ is the orthogonal complement with respect to the $H(\mathbf{curl}\,)$-inner product). First show that $V \subset W$ (using that $\boldsymbol{\gamma}_T^\times$ is continuous). Secondly show that $V^\perp \cap W = \{0\}$. For the latter, calculate $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}$ for $\boldsymbol{v} \in V^\perp \cap W$ and then $\|\boldsymbol{v}\|^2_{H(\mathbf{curl}\,)}$ using Theorem 2.28 (iv).

**15** Let $\Omega \subset \mathbb{R}^2$. We define the following 2-dimensional vector and scalar curl:

$$\begin{aligned} \mathbf{curl}\,\varphi &= \begin{pmatrix} -\frac{\partial\varphi}{\partial x_2} \\ \frac{\partial\varphi}{\partial x_1} \end{pmatrix} \qquad \text{for } \varphi \in C^1(\bar\Omega), \\ \mathrm{curl}\,\boldsymbol{v} &= \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \qquad \text{for } \boldsymbol{v} \in \boldsymbol{C}^1(\bar\Omega). \end{aligned}$$

(a) Find the definition of the corresponding weak derivatives and of the (vector-valued) space $H(\mathrm{curl}, \Omega)$.

(b) Show the following two De Rham complexes:

$$\begin{aligned} &H^1(\Omega) \xrightarrow{\nabla} H(\mathrm{curl}, \Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega), \\ &H^1(\Omega) \xrightarrow{\mathbf{curl}} H(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega). \end{aligned}$$