10 Let $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ and $\boldsymbol{v}_1 \in [C^1(\overline{\Omega}_1)]^3$, $\boldsymbol{v}_2 \in [C^1(\overline{\Omega}_2)]^3$. We define

$$\boldsymbol{v}(x) := \left\{ \begin{array}{ll} \boldsymbol{v}_1(x), & x \in \Omega_1 \\ \boldsymbol{v}_2(x), & x \in \Omega_2 \end{array} \right\} \in [L^2(\Omega)]^3.$$

Show that

$$\boldsymbol{v} \in H(\mathbf{curl}\,, \Omega) \quad \Longleftrightarrow \quad \boldsymbol{v}_1 \times \boldsymbol{n} = \boldsymbol{v}_2 \times \boldsymbol{n} \text{ on } \Gamma. \tag{4.1}$$

11 Consider the smoothing operators S_g^ε from the lecture. Show that

$$\exists\, C > 0 \quad \forall\, \varepsilon \in (0, \varepsilon_0) \quad \forall\, w \in L^2(\Omega) \,:\, \|S_g^\varepsilon\, w\|_{L^2(\Omega)} \,\leq\, C\|w\|_{L^2(\Omega)}. \tag{4.2}$$

Hint: Use the definition of S_g^ε, exchange $\int_\Omega$ and $\int_{B(0,1)}$, transform Ω to $\phi^\varepsilon(\Omega)$ and use Lemma 2.10 from the lecture to bound the Jacobi determinant.

12 Let $\hat{T}$, T be tetrahedra, $\phi : \hat{T} \to T$ an affine linear bijective map, $F = \phi'$, $J = \det F = \frac{|T|}{|\hat{T}|} > 0$. Let $\hat{f} \subset \partial\hat{T}$ be a (flat) face with normal $\hat{\boldsymbol{n}}$ and let $f = \phi(\hat{f})$ be the corresponding face of T with normal $\boldsymbol{n}$.

Show that for $\hat{\boldsymbol{v}}, \hat{\boldsymbol{w}} \in [C^1(\overline{\hat{T}})]^3$:

$$\int_f (\boldsymbol{v} \times \boldsymbol{n}) \cdot \boldsymbol{w}\, d\sigma = \int_{\hat{f}} (\hat{\boldsymbol{v}} \times \hat{\boldsymbol{n}}) \cdot \hat{\boldsymbol{w}}\, d\hat{\sigma}, \tag{4.3}$$

where $\boldsymbol{v}$ and $\boldsymbol{w}$ are the covariant transformations of $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{w}}$, e.g. $\boldsymbol{v} \circ \phi = F^{-T}\hat{\boldsymbol{v}}$.

Hint: Use that

$$\boldsymbol{n} = \left(\frac{J}{J_f}\, F^{-T}\hat{\boldsymbol{n}} \right) \circ \phi^{-1} \text{ with } J_f = \frac{|f|}{|\hat{f}|}$$

and that $(A\,\boldsymbol{y}) \times (A\,\boldsymbol{z}) = (\det A)A^{-T}(\boldsymbol{y} \times \boldsymbol{z})$ for $A \in \mathbb{R}^{3\times 3}$, $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^3$.