

# T U T O R I A L

## “Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

### **Tutorial 13**

Tuesday, 21 June 2011, Time: 10<sup>15</sup> – 11<sup>45</sup>, Room: SR / T 642.

### 3.9 Clément’s Interpolator

- 63** Let  $\Omega = (0, 1)$  and consider the equidistant subdivision into elements  $[x_{i-1}, x_i] = [(i-1)h, ih]$ ,  $i = 1, \dots, n$ . For each node  $x_i = ih$ ,  $i = 1, \dots, n-1$  we define the local  $L_2$ -projection  $P_i : L_2(x_{i-1}, x_{i+1}) \rightarrow \mathcal{P}_0(x_{i-1}, x_{i+1}) = \mathbb{R}$  by

$$\int_{x_{i-1}}^{x_{i+1}} (P_i v) q \, dx = \int_{x_{i-1}}^{x_{i+1}} v q \, dx \quad \forall q \in \mathcal{P}_0(x_{i-1}, x_{i+1}) \quad \forall v \in L_2(x_{i-1}, x_{i+1}),$$

where  $\mathcal{P}_0(x_{i-1}, x_{i+1})$  are the constant functions on  $(x_{i-1}, x_{i+1})$ . Show that

- 1)  $P_i v = \frac{1}{2h} \int_{x_{i-1}}^{x_{i+1}} v(x) \, dx$ ,
  - 2)  $\|v - P_i v\|_{L_2(x_{i-1}, x_{i+1})} \leq ch \|v'\|_{L_2(x_{i-1}, x_{i+1})} \quad \forall v \in H^1(x_{i-1}, x_{i+1})$ .
- 64** Let  $V_0 := H_0^1(0, 1)$  and  $V_{0h} := \text{span}\{p^{(j)} : j = 1, \dots, n-1\}$  where  $p^{(j)}$  is the nodal basis function associated to the node  $x_j$ . We define Clément’s interpolator  $I_h : L_2(0, 1) \rightarrow V_{0h} \subset V_0$  by

$$(I_h u)(x) := \sum_{j=1}^{n-1} (P_j u) p^{(j)}(x) \quad \text{for } x \in [0, 1].$$

Show that  $\|u - I_h u\|_{L_2(0,1)} \leq ch \|u'\|_{L_2(0,1)} \quad \forall u \in V_0$ .

*Hint:* Follow your lecture notes. The difference here is that we have to treat homogeneous Dirichlet boundary conditions ! Show and use the scaled Friedrichs inequality

$$\|u\|_{L_2(x_0, x_1)} \leq c_F h |u|_{H^1(x_0, x_1)},$$

with  $c_F \neq c_F(h)$ . The same inequality is valid for the interval  $(x_{n-1}, x_n)$ .

- 65** Show that  $|u - I_h u|_{H^1(0,1)} \leq c \|u\|_{H^1(0,1)}$ .

### 3.10 A posteriori error estimates

- 66** In Section 3.6.2 of our lecture, we derived the residual error estimator for the Dirichlet problem of Poisson’s equation. How do we have to modify this estimator such that it works for our model problem CHIP ? Derive the right residual error estimator for the CHIP problem !