

# TUTORIAL

## “Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

### **Tutorial 12**

Thursday, 9 June 2011, Time: 10<sup>15</sup> – 11<sup>45</sup>, Room: SR / T 111.

### 3.8 Variational Crimes

Consider the one-dimensional BVP to find  $u \in V_g = V_0 = H_0^1(0, 1)$ :

$$\int_0^1 \lambda(x) u'(x) v'(x) dx = \int_0^1 f(x) v(x) dx \quad \forall v \in V_0, \quad (3.35)$$

where  $f \in L^2(0, 1)$  and  $\lambda \in L^\infty(0, 1)$ . We assume that there exists positive constants  $\underline{\lambda}$  and  $\bar{\lambda}$  such that  $0 < \underline{\lambda} \leq \lambda(x) \leq \bar{\lambda} \quad \forall x \in (0, 1)$  a.e. Let us consider a finite element discretization with continuous linear finite elements ( $\mathcal{F}(\Delta) = \mathcal{P}_1$ ) on an equidistant grid ( $x_i = ih, i = \bar{0}, n+1, h = 1/(n+1)$ ),  $\delta_r = (x_{r-1}, x_r), r = \bar{1}, n+1$ ). Now we approximate the bilinear form  $a(\cdot, \cdot)$  and the linear form  $\langle F, \cdot \rangle$  defined in (3.35) using numerical integration, namely the *midpoint rule*:

$$a_h(u_h, v_h) = \sum_{r=1}^{n+1} h \lambda(x_r^*) u'_h(x_r^*) v'_h(x_r^*) \quad \text{and} \quad \langle F_h, v_h \rangle_h = \sum_{r=1}^{n+1} h f(x_r^*) v_h(x_r^*), \quad (3.36)$$

where  $x_r^* = x_{\delta_r}(\frac{1}{2}) = x_{r-1} + \frac{1}{2}h$ . To ensure that these expressions are well-defined we assume (for simplicity) that

$$\lambda, f \in W_\infty^1(0, 1).$$

Let  $\tilde{u}_h \in V_{0h}$  be such that  $a_h(\tilde{u}_h, v_h) = \langle F_h, v_h \rangle_h$  for all  $v_h \in V_{0h}$ . We are interested whether the error  $\|u - \tilde{u}_h\|_{H^1(0,1)}$  obeys the same asymptotics with respect to  $h$  than if we compute  $a(\cdot, \cdot)$  and  $\langle F, \cdot \rangle$  exactly. This investigation will be done using Strang's first lemma. Throughout this tutorial, we choose  $\|\cdot\|_{V_0} := |\cdot|_{H^1(0,1)}$  that is a norm in  $V_0 = H_0^1(0, 1)$ .

**59** Show that the bilinear and linear forms above fulfill the standard assumptions (33) and the additional assumption (34) from the lecture notes. The latter is called *uniform ellipticity* of the discrete bilinear form  $a_h(\cdot, \cdot)$ , i.e. there exists a positive constant  $\mu_3 \neq \mu_3(h)$  such that

$$a_h(v_h, v_h) \geq \mu_3 \|v_h\|_{V_0}^2 \quad \forall v_h \in V_{0h}.$$

*Hint:* For the uniform ellipticity, use that  $\lambda \geq \underline{\lambda}$  and that the midpoint rule is exact for some (which?) polynomials.

We can now apply Strang's first lemma which yields

$$\|u - \tilde{u}_h\|_{V_0} \leq c \left\{ \inf_{v_h \in V_0} \left[ \|u - v_h\|_{V_0} + \sup_{w_h \in V_{0h}} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_{V_0}} \right] + \right. \quad (3.37)$$

$$\left. + \sup_{w_h \in V_{0h}} \frac{|\langle F, w_h \rangle - \langle F_h, w_h \rangle_h|}{\|w_h\|_{V_0}} \right\}$$

**60** For  $\varphi \in H^1(\delta_r)$ , prove that

$$\left| \int_{\delta_r} \varphi(x) dx - h \varphi(x_r^*) \right| \leq c h |\varphi|_{H^1(\delta_r)},$$

similarly to the exercises in Tutorial 07. Then set  $\varphi = f w_h$  and show that

$$|\langle F, w_h \rangle - \langle F_h, w_h \rangle_h| \leq c h \|f\|_{W^{1,\infty}(0,1)} \|w_h\|_{V_0}.$$

**61** For  $v_h, w_h \in V_{0h}$ , prove that

$$|a(v_h, w_h) - a_h(v_h, w_h)| \leq c h |\lambda|_{W_\infty^1(0,1)} \|v_h\|_{V_0} \|w_h\|_{V_0}.$$

*Hint:* Treat each element separately and use that  $v'_h, w'_h$  are constant on each element, so that we are left with  $|\int_{\delta_r} \lambda(x) dx - h \lambda(x_r^*)|$ . To get an error bound for this term, use Bramble-Hilbert on the reference element.

**62** Show that if  $u \in H^2(0, 1)$  then

$$\|u - \tilde{u}_h\|_{V_0} \leq c h \left\{ |u|_{H^2(0,1)} + |\lambda|_{W_\infty^1(0,1)} \|u\|_{V_0} + \|f\|_{W_\infty^1(0,1)} \right\}.$$

*Hint:* Choose  $v_h = u_h$  in (3.37), where  $u_h \in V_{0h}$  is the finite element solution in the exact case, i.e.

$$a(u_h, v_h) = \langle F, v_h \rangle \quad \forall v_h \in V_{0h}.$$

Show and use that

$$\|u_h\|_{V_0} \leq c(\lambda) \|u\|_{V_0}.$$