T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 09 Tuesday, 17 May 2011, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: SR / T 642.

3.5 Properties of the Finite Elements Equations

42 Prove that the inheritance identity

$$(K_h \underline{u}_h, \underline{v}_h) \;=\; a(u_h, v_h) \qquad \forall \underline{u}_h, \underline{v}_h \leftrightarrow u_h, v_h \in V_{0h} \; ! \tag{3.22}$$

is valid !

43 Show that the eigenvalue estimates in Theorem 2.4 are sharp with respect to the h-order by proving the following statement. There exist positive constants $\underline{c}'_E$ and $\overline{c}'_E$ independent of h satisfying the estimates

$$\lambda_{\min}(K_h) \;\le\; \underline{c}'_E h^d \qquad \text{and} \qquad \lambda_{\max}(K_h) \;\ge\; \overline{c}'_E h^{d-2} \,. \tag{3.23}$$

For simplicity, consider the 1D case ($d = 1$):

$$\begin{aligned} -u''(x) &= f(x) \qquad \forall x \in (0,1)\,, \\ u(0) = u(1) &= 0\,. \end{aligned}$$

44 Show that, for a regular triagulation according to Definition 2.3, there exist h-independent positive constants $\underline{c}_0$ and $\overline{c}_0$ satisfying the inequalities

$$\underline{c}_0 h^d (\underline{v}_h, \underline{v}_h) \;\le\; (M_h \underline{v}_h, \underline{v}_h) \;\le\; \overline{c}_0 h^d (\underline{v}_h, \underline{v}_h) \tag{3.24}$$

for all $\underline{v}_h \in \mathbb{R}^{N_h}$, where M_h denotes the mass-matrix defined by the identity

$$(M_h \underline{u}_h, \underline{v}_h) := \int_\Omega u_h(x) v_h(x) dx \qquad \forall \underline{u}_h, \underline{v}_h \leftrightarrow u_h, v_h \in V_{0h} \tag{3.25}$$

The spectral inequalities (3.25) yield that the mass matrix M_h is well conditioned, i.e the spectral condition number $\text{cond}_2(M_h)$ can be bounded by the h-independent constant $\overline{c}_0 / \underline{c}_0$.

45* Let $\lambda = \lambda_{\max}$ be the maximal eigenvalue of the generalized eigenvalue problem

$$K_h \underline{u}_h = \lambda M_h \underline{u}_h \tag{3.26}$$

and let $\lambda_r = \lambda_{r,\max}$ be the maximal eigenvalues of generalized eigenvalue problems

$$K_h^{(r)} \underline{u}_h^{(r)} = \lambda_r M_h^{(r)} \underline{u}_h^{(r)}, \tag{3.27}$$

where $K_h^{(r)}$ and $M_h^{(r)}$ denote the (local) element stiffness and mass matrices for element number $r = 1, 2, ..., R_h$, i. e., it holds

$$K_h = \sum_{r=1}^{R_h} C_r K_h^{(r)} C_r^T \qquad \text{and} \qquad M_h = \sum_{r=1}^{R_h} C_r M_h^{(r)} C_r^T .$$

Show the eigenvalue estimate

$$\lambda \le \max_{r=1,2,...,R_h} \lambda_r . \tag{3.28}$$