T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 07 Tuesday, 3 May 2011, Time: $10^{\underline{15}} - 11^{\underline{45}}$, Room: SR / T 642.

3.4 Generation of systems of Finite Elements Equations

29 Show that the integration rule

$$\int_{\Delta} f(\xi,\eta)d\xi d\eta \approx \frac{1}{2}\{\alpha_1 f(\xi_1,\eta_1) + \alpha_2 f(\xi_2,\eta_2) + \alpha_3 f(\xi_3,\eta_3)\} \tag{3.15}$$

integrates quadratic polynomials exactly, if the the weights α_i and the integration points (ξ_i, η_i) are choosen as follows: $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$ und $(\xi_1, \eta_1) = (1/2, 0)$, $(\xi_2, \eta_2) = (1/2, 1/2)$, $(\xi_3, \eta_3) = (0, 1/2)$.

Hint: cf. also Exercise 20 !

30 Let us assume that $\mathcal{T}_h = \{\delta_r : r \in \mathbb{R}_h\}$ is a regular triangulation of the polygonally bounded Lipschitz domain $\overline{\Omega} = \cup_{r\in\mathbb{R}_h}\overline{\delta}_r \subset \mathbb{R}^2$ into triangles δ_r, and let $u \in H^2(\Omega)$. Let us now compute the integral

$$I(u) = \int_{\Omega} u(x)dx$$

by the quadrature rule

$$I_h(u) = \sum_{r\in\mathbb{R}_h} u(x_{\delta_r}(\xi^*))\,|\delta_r|,$$

where $x_{\delta_r}(\cdot)$ maps the unit triangle Δ onto δ_r, and $\xi^* = (1/3, 1/3)$. Show that

$$|I(u) - I_h(u)| \leq ch^2|u|_{H^2(\Omega)},$$

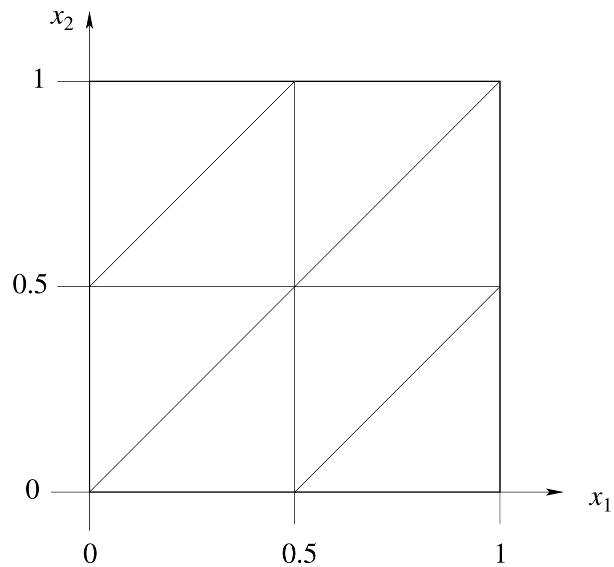
where c is some generic positive constant. Can you weaken the assumption that $u \in H^2(\Omega)$?

Hint: Use the mapping principle and the Bramble-Hilbert Lemma; cf. also Exercise 20 !

31 Generate the system of finite element equations for the mixed boundary value problem

$$
\begin{aligned}
-\Delta u(x_1, x_2) &= 0 \quad \forall (x_1, x_2) \in \Omega := (0,1) \times (0,1), && (3.16)\\
u(x_1, 1) &= 0 \quad \forall x_1 \in [0,1], && (3.17)\\
u(1, x_2) &= 0 \quad \forall x_2 \in [0,1], && (3.18)\\
u_{x_1}(0, x_2) &= x_2 - 1 \quad \forall x_2 \in (0,1], && (3.19)\\
u_{x_2}(x_1, 0) &= x_1 - 1 \quad \forall x_1 \in (0,1), && (3.20)
\end{aligned}
$$

and for the triangulation shown in the attached figure. Solve this linear system of algebraic equations ! Note that u_{x_1} and u_{x_2} denote the partial derivatives with respect to x_1 and x_2.



Programming

Reference element

In this and the next tutorials we consider Courant's finite element. The reference triangle is given by

$$\Delta = \{\xi \in \mathbb{R}^2 : \xi_1 \geq 0,\ \xi_2 \geq 0,\ \xi_1 + \xi_2 \leq 1\},$$

with vertices $\xi^{(0)} = (0,0)$, $\xi^{(1)} = (1,0)$, and $\xi^{(2)} = (0,1)$, the space of shape functions is P_1, and the nodal variables are the evaluations at the three vertices. Recall that the nodal shape functions are given by

$$\begin{aligned} p^{(0)}(\xi) &= 1 - \xi_1 - \xi_2, \\ p^{(1)}(\xi) &= \xi_1, \\ p^{(2)}(\xi) &= \xi_2. \end{aligned}$$

To model *small* vectors from $\mathbb{R}^n$ and $n \times m$ matrices, where $m,\ n \in \{2,3\}$, I recommend to use `vec.hh` and `mat.hh` (see also the demo `matvecdemo.cc`). There 0-based indices are used throughout, for example:

$$\xi \in \mathbb{R}^2 \leftrightarrow \texttt{Vec<2> xi} \qquad \begin{aligned} \xi_1 &\leftrightarrow \texttt{xi[0]} \\ \xi_2 &\leftrightarrow \texttt{xi[1]} \end{aligned}$$

32 Write two functions

```
double calcShape (int i, const Vec<2>& xi);
Vec<2> calcDShape (int i, const Vec<2>& xi);
```

that compute the *value* $p^{(\alpha)}(\xi)$ and the *gradient* $\nabla_\xi p^{(\alpha)}(\xi)$ of a nodal shape function, respectively, where `xi`$=\xi$ and `i`$=\alpha$.